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7/25/68

VIBRATION OF LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

A THESIS

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VIBRATION OF LONGITUDINALLY STIFFENED

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This thesis is dedicated to Cindy and Sherry.

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LIST OF PRINCIPAL SYMBOLS

a	Radius of shell
A_{mn}, B_{mn}, C_{mn}	Symmetric Fourier coefficients
$A'_{mn}, B'_{mn}, C'_{mn}$	Antisymmetric Fourier coefficients
A_r	Cross-sectional area of the rth stringer
(a_{yr}, a_{zr})	Coordinates of the shear center with respect to the centroid
C	Centroid of stringer
C_{sn}, C'_{sn}	Arbitrary constants in complementary solution for the shell
C_{wr}	Warping constant for the rth stringer
$[D]$	Coefficient matrix
$e_{xx}, e_{\phi\phi}, \gamma_{x\phi}$	Middle surface strain components in shell
E	Young's modulus for shell
E_r^*	Reduced Young's modulus for rth stringer
$(GJ)_r$	Torsional stiffness of the rth stringer
h	Shell thickness

$I_{y'r}, I_{z'r}, I_{y'z'r}$	Products of inertia with respect to the centroid of the rth stringer
I_{or}	Polar moment of inertia of rth stringer
J	St. Venant constant for uniform torsion
k_{ij}	Submatrices of stiffness matrix
K	Total number of stringers
$\kappa_x, \kappa_\phi, \kappa_{x\phi}$	Change of curvatures of middle surface of the shell
ℓ	Length of shell or stringers
m	Number of axial half-waves
M	Maximum value taken on by m
m_{ij}	Submatrices of mass matrix
n	Number of circumferential waves
N	Maximum value taken on by n
ν, ν_r	Poisson's ratio for shell and stringer, respectively
O	Shear center of stringer
P_x, P_ϕ, P_z	Surface loads on shell in the x, ϕ, z directions, respectively
$\{q\}$	Column matrix of generalized coordinates
t	Time
T	Kinetic energy

u, v, w	Longitudinal, tangential, and radial middle surface displacements of the shell
V	Potential energy
x, y, z	Stringer fixed coordinates
X, Y, Z	Shell fixed coordinates
ζ_r, ξ_r, η_r	Longitudinal, tangential, and radial displacements of the rth stringer
Θ_{sr}	Angle of twist of the rth stringer
ϕ	Angular coordinate of the shell
ϕ_r	Angular coordinate of the rth stringer
ρ, ρ_r	Shell and stringer density, respectively
ω	Circular frequency of composite shell
Ω	Frequency parameter
β	Shell stiffness parameter
δ_{ij}	Kronecker delta

SUMMARY

This study is concerned with a theoretical analysis of the free vibration characteristics of a thin cylindrical shell reinforced by longitudinal ribs for any set of admissible boundary conditions. The usual assumption that the stringers are closely spaced is not made and hence the stiffened shell is not treated as an orthotropic continuum but rather as an isotropic cylinder interacting with a set of discrete thin-walled beams.

A new method of analyzing this type of structure is presented in which the shell-stringer interaction loads are represented by double Fourier series with unknown coefficients. Once the analysis of the shell and stringer problems has been completed, kinematical conditions at the line of attachment that require compatible deformations between the shell and stringers are enforced. This yields a linear eigenvalue problem in the unknown loading coefficients for determination of the eigenfrequencies of the composite structure.

The mathematical model which is developed is quite flexible. Since the number of stringers, and their spacing is left arbitrary, there are no restrictions due to smearing or averaging the stringer effects over the surface of the shell. Furthermore, the governing differential equations of the composite structure are satisfied as are all of the boundary conditions.

The Ritz method is employed in the analysis to compare the simplified Donnell's shell theory with Flügge's more exact theory for a stringer stiffened shell with simply supported boundary conditions. It was found that these theories were in good agreement with each other and with the experimental results found in the literature. It was also found that neglecting rotatory inertia of the shell and stringers was a justifiable assumption for the lower axial modes ($m \leq 4$). However, the assumption of neglecting in-plane inertias for the shell in calculating the radial frequencies can lead to results in serious disagreement with theoretical calculations based on retaining these terms. The analysis also shows that the presence of even a few stringers on a shell may substantially couple the circumferential modes when one considers the stringers as discrete structural elements. Also, the frequencies associated with symmetric and antisymmetric circumferential modes can have different values, which is not true for the unstiffened shell.

A general linear theory is developed based upon solving the coupled governing equations of motion for the shell and stringers. Donnell's equations are employed for the shell and V. Z. Vlasov's equations are used for the stringers. A particular numerical example is considered to illustrate the application of the general analysis to a realistic stiffened shell. The boundary conditions are chosen to be clamped-clamped. Also, results of the general analysis are compared with the Ritz analysis for the simply supported case and with some experimental data in the literature. Good agreement between mode shapes and frequencies was noted for this case.

CHAPTER I

INTRODUCTION

Since reinforced cylindrical shells are used extensively in the aerospace, nuclear, marine, and petrochemical industries as structural components in missiles, aircraft, submarines, nuclear reactor vessels, and refinery equipment, investigations on the free vibration characteristics of these shells continues to be of interest to researchers in the field of solid mechanics. The stiffening of these shells is accomplished by attaching longitudinal ribs and/or ring frames to the shell surface for the purpose of increasing the strength and stability of the shell while still retaining a relatively lightweight composite structure.

In the past, the usual method of analyzing reinforced cylindrical shells was to treat the structure as an orthotropic continuum with effective extensional and flexural stiffness by considering the stiffeners to be closely spaced and to average the stiffening effects over the surface of the shell as was done by Hoppman [1]* and Miller [2]. This method was used by McElman, Mickulas, and Stein [3] to investigate the effects of stringer eccentricity (whether the stiffener center of gravity is inside or outside of the shell wall) on stiffened flat plates and cylindrical shells. The conclusion reached from this

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*The numbers enclosed in brackets denote the correspondingly numbered reference in the Literature Cited.

investigation was that stiffener eccentricity effects are important and should be considered in the analysis of stiffened plates and cylindrical shells. However, they found that since eccentricity effects depend on the configuration and physical properties of both the cylinder and the stiffeners, it was rather difficult to make any concrete statements regarding stringer eccentricity.

An alternative approach in analyzing reinforced shells is to treat the stiffeners as discrete structural components. The advantage of this approach is that the stiffeners need not be closely spaced, identical in size, and may be few in number. The discrete approach has recently been adopted by several authors in investigating the vibration of stiffened cylindrical shells. Of the investigations using the discrete approach, references [4,5] deal only with longitudinal ribs and reference [6] considers both rings and stringers. Reference [4] considers the free vibration of stringer stiffened cylinders by using the orthotropic theory and the method of transfer matrices and compares these theories with experimental results for simply supported cylinders with four or eight stringers. The Rayleigh-Ritz energy method is employed in the other given references and with the exception of Egle and Sewall [6] only simply supported boundary conditions are considered. Ojalvo and Newman [5] were concerned with the effect of an attached mass point on the frequencies of a simply supported cylindrical shell with many equally spaced stringers. Egle and Sewall [6] use the energy method but leave the axial mode unspecified in their expression for the assumed displacement functions. The authors suggest that beam-type axial modes be used to allow for various end conditions. Numerical results are presented for a simply supported, stringer stiffened cylindrical shell using Donnell's shell theory.

One of the disadvantages of using an approximate energy method is in the selection of the approximating functions. This depends on the user's intuition and experience and there is no systematic way to arrive at a best choice for an approximating sequence. The application of the Ritz method requires that the approximating functions be members of a complete system of functions and this is one of the basic difficulties connected with the method. Often it is difficult to find a complete system of coordinate functions satisfying the geometric boundary conditions of a problem. Then, too, since the coordinate functions do not in general satisfy the governing differential equations of motion of a given continuum it is sometimes difficult to assess the rate at which an approximate solution is converging to the exact solution. Often one must look for reassurances in making successive approximations to lend confidence to the results and where possible to compare the results with established information.

To investigate the problem of free vibration of a stiffened cylinder with arbitrary boundary conditions, an approach is adopted by which the coupled equations of motion are solved. This avoids one of the major disadvantages of the energy method in that one is not faced with the problem of having to select some set of approximating functions. Furthermore, one can satisfy the governing equations of motion of the composite structure and all of the boundary conditions exactly.

However, one is required to make a representation of the unknown forces of constraint which is avoided in an energy approach.

In proposing a general linear theory, it is very desirable to use as simplified a shell theory as possible in order to minimize the mathematical difficulties attendant with such an approach. To this end, an energy approach using a convenient set of coordinate functions can be useful in comparing the simplified Donnell's shell theory to the more exact Flügge's theory to see if they are in reasonable agreement for stiffened shells.

Three qualifications that are desirable in a mathematical model of an elastic system are (1) the model should adequately represent the important characteristics of the system, (2) the resulting mathematical equations describing the model's behavior should not be so unwieldly as to preclude a reasonable attack upon the problem, and (3) the actual results of the model should be reasonably accurate.

The energy method satisfies the first and second requirements for a good model. The results of the model are compared with existing experimental data to ascertain if the third criterion is satisfied. Furthermore, the energy method has been used in investigating the effect of the shear center not coinciding with the centroid, and the nonsymmetrical property (cross product of inertia) which is important for nonsymmetrical sections. From a differential equations point of view, the partial differential equations governing the motion of the stringer will be uncoupled if the shear center coincides with the centroid (or if such an effect can be considered negligible) and if the cross product of inertia is zero (or can be neglected). Therefore, the energy method when used with convenient coordinate functions can be useful in furnishing one with a qualitative feel as well as explicit justification for various assumptions which one may postulate in attempting to formulate a tractable set of partial differential equations which govern the behavior of a stiffened cylindrical shell. Scruggs and Pierce [7] employed the energy method to investigate the free vibration of a ring and stringer stiffened shell for simply supported boundary conditions and obtained good agreement for the modes and frequencies investigated. The author, however, did not employ series solutions to represent the circumferential mode shape.

The purpose of this work is to develop a mathematical model for investigating the free vibration characteristics of a thin cylindrical shell reinforced by discrete stringers with arbitrary admissible boundary conditions by considering a thin monocoque cylinder interacting with a system of longitudinal ribs and using the energy method as a springboard to develop such a model.

Numerical examples are presented for both the Ritz analysis and the analysis by solving the coupled governing differential equations of motion.

CHAPTER II

TOTAL ENERGY FOR SIMPLY SUPPORTED LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

The Ritz method is employed in the present analysis to determine the natural frequencies and mode shapes of thin circular cylindrical shells stiffened by discrete elastic stringers with simply supported boundary conditions. Simply supported boundary conditions are chosen because they enable one to express the shell displacements in terms of a linearly independent, complete set of functions which satisfies all of the boundary conditions for the shell and stringers. Also, the existing experimental data for free vibration of stringer stiffened cylinders is only for simply supported boundary conditions. Therefore, this choice of boundary conditions enables one to adopt the Ritz method under extremely favorable conditions.

The numerical results of this method will be used later for comparison purposes with the results of the general analysis as given in Chapter IV. This will furnish an explicit comparison of two different approaches to the problem of free vibration of stiffened cylinders. If the results of these two methods are in good agreement, then this will furnish a confirmation of the derivation and programming of the general analysis.

The analysis incorporates rotatory inertia, bending, extension, and torsion of the stringers and does not neglect warping, transverse stiffening effects, rotatory inertia of the shell, nor is it assumed that the shear center of the stringers coincides with the centroid nor does it neglect in-plane inertias in the numerical solution of the problem as is done in reference [6]. The general procedure for the Ritz method is given in the following steps.

The kinetic and potential energy functionals are constructed for the cylindrical shell and the stringers. These expressions are combined to give the Lagrangian for the stiffened cylinder in terms of the middle surface displacements of the shell by relating the displacements of the stringers to the middle surface displacements of the shell. Next, the displacements of the cylinder are assumed in the form of finite Fourier series with undetermined coefficients where each term of the series satisfies simply supported boundary conditions. The assumed displacements are substituted into the energy functionals and the Euler-Lagrange equations are utilized to yield a linear eigenvalue problem in the unknown amplitudes. The eigenvalue problem is then solved numerically for the approximate frequencies and mode shapes of the stiffened cylinder.

Potential Energy of System

The following assumptions concerning the shell are made:

(i) The shell material is isotropic and homogeneous.

(ii) The linear strain-displacement relations are employed.

(iii) The thickness of the shell is small compared with the smallest radius of curvature of its middle surface.

(iv) The stress components normal to the middle surface are small compared with the other stress components and may be neglected.

(v) The normals to the unstrained middle surface deform into the normals of the strained middle surface.

(vi) The normals to the strained middle surface undergo no elongation.

The assumptions (v) and (vi) together are commonly called the Kirchhoff-Love hypotheses.

The displacements u, v, and w of the middle surface of the shell are assumed to be given by the following finite Fourier series in ϕ and x (see Figure 1):

$$u(x,\phi,t) = \sum_{m=o}^{M} \sum_{n=o}^{N} [A_{mn}(t)\cos(n\phi) + A'_{mn}(t)\sin(n\phi)]\cos \frac{m\pi x}{\ell} ,$$

$$v(x,\phi,t) = \sum_{m=1}^{M} \sum_{n=o}^{N} [B_{mn}(t)\sin(n\phi) - B'_{mn}(t)\cos(n\phi)]\sin \frac{m\pi x}{\ell} ,$$

and

$$w(x,\phi,t) = \sum_{m=1}^{M} \sum_{n=o}^{N} [C_{mn}(t)\cos(n\phi) + C'_{mn}\sin(n\phi)]\sin \frac{m\pi x}{\ell} . \quad (2.1)$$

Each term of the series in Equations (2.1) satisfies the boundary conditions:

$$v = w = N_x = M_x = 0 \text{ at } x = 0,\ell \quad (2.2)$$

where N_x is the longitudinal stress resultant and M_x is the bending moment (see Appendix B). These are simply supported boundary conditions without axial constraint and are sometimes called "freely supported" in the literature. The unprimed coefficients (A_{mn},B_{mn},C_{mn}) are associated with the symmetric circumferential modes (i.e., modes which are symmetric with respect to the plane ϕ=0) and the prime coefficients (A'_{mn}, B'_{mn},C'_{mn}) are associated with the antisymmetric circumferential modes (i.e., modes which are antisymmetric with respect to the plane ϕ = 0). It should be noted that the mnth term for u, v, and w represents the modal functions of an unstiffened thin cylindrical shell with simply supported boundary conditions.

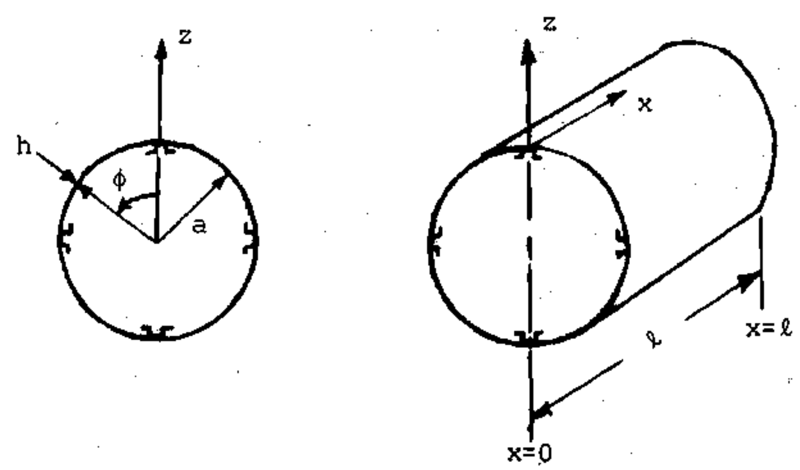


Figure 1. Stiffened Shell Coordinate System

The strain-displacement relations of the middle surface of a cylindrical shell for the coordinate system depicted in Figure 1 are given by Flügge [8] as:

$$e_{xx} = \frac{\partial u}{\partial x} , \qquad (2.3)$$

$$e_{\phi\phi} = \frac{1}{a}\frac{\partial v}{\partial \phi} + \frac{w}{a} ,$$

$$\gamma_{x\phi} = \frac{1}{a}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial x} .$$

The changes of curvature for a thin cylindrical shell are given by Flügge [8] as:

$$\kappa_x = \frac{\partial^2 w}{\partial x^2} , \qquad (2.4)$$

$$\kappa_\phi = \frac{1}{a^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{w}{a^2} ,$$

$$\kappa_{x\phi} = \frac{1}{2a^2}\frac{\partial u}{\partial \phi} - \frac{1}{2a}\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial^2 w}{\partial x \partial \phi} .$$

Substituting the expressions for the assumed displacements into Equations (2.3) and (2.4), the strains and changes of curvature of the shell become:

$$e_{xx} = - \sum_{m=1}^{M} \sum_{n=o}^{N} (A_{mn}\cos(n\phi) + A'_{mn}\sin(n\phi))\alpha_m \sin(\alpha_m x), \qquad (2.5)$$

$$e_{\phi\phi} = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=o}^{N} [(B_{mn}(n) + C_{mn})\cos(n\phi) + (B'_{mn}(n) + C'_{mn})\sin(n\phi)]\sin(\alpha_m x),$$

$$\gamma_{x\phi} = \sum_{m=1}^{M} \sum_{n=0}^{N} [(B_{mn} - (n)A_{mn})\sin(n\phi) + ((n)A'_{mn} - B'_{mn})\alpha_m \cos(n\phi)]\cos(\alpha_m x),$$

$$\kappa_x = -\sum_{m=1}^{M} \sum_{n=0}^{N} (C_{mn}\cos(n\phi) + C'_{mn}\sin(n\phi))\alpha_m^2 \sin(\alpha_m x),$$

$$\kappa_\phi = \frac{1}{a^2} \sum_{m=1}^{M} \sum_{n=0}^{N} (C_{mn}(1-n^2)\cos(n\phi) + C'_{mn}(1-n^2)\sin(n\phi))\sin(\alpha_m x),$$

$$\kappa_{x\phi} = \frac{1}{a^2} \sum_{m=1}^{M} \sum_{n=0}^{N} \left\{ \left(\frac{n}{2a^2} A'_{mn} + \frac{\alpha_m}{2a} B'_{mn} + \frac{(\alpha_m)(n)}{a} C'_{mn} \right) \cos(n\phi) - \left(\frac{n}{2a^2} A_{mn} + \frac{\alpha_m}{2a} B_{mn} + \frac{\alpha_m (n)}{a} C_{mn} \right) \sin(n\phi) \right\} \cos(\alpha_m x)$$

where

$$\alpha_m = \frac{m\pi}{\ell} .$$

The strain energy of a thin cylindrical shell is given by Love [9] as:

$$V_{SHELL} = \frac{Eha}{2(1-\nu^2)} \int_0^{\ell} \int_0^{2\pi} [(e_{xx}+e_{\phi\phi})^2 - 2(1-\nu)(e_{xx}e_{\phi\phi} - \frac{1}{4}\gamma_{x\phi}^2)]dxd\phi +$$

$$\frac{Eh^3a}{24(1-\nu^2)} \int_0^{\ell} \int_0^{2\pi} [(\kappa_x+\kappa_\phi)^2 - 2(1-\nu)(\kappa_x\kappa_\phi - \kappa_{x\phi}^2)]dxd\phi \qquad (2.6)$$

when the first term represents the membrane energy associated with the stretching of the shell middle surface, and the second term is the strain energy of bending.

Inserting Equations (2.5) into (2.6) and integrating yields:

$$V_{SHELL} = \frac{\pi Eha\ell}{4(1-\nu^2)} \sum_{m=1}^{M} \sum_{n=0}^{N} \{(-\alpha_m A_{mn} + \frac{n}{a} B_{mn} + \frac{1}{a} C_{mn})^2 + (-\alpha_m A'_{mn} + \frac{n}{a} B'_{mn} + \frac{1}{a} C'_{mn})^2$$

$$+ 2(1-\nu)[\alpha_m A_{mn}(\frac{n}{a} B_{mn} + \frac{1}{a} C_{mn}) + \frac{1}{4}(\alpha_m B_{mn} - \frac{n}{a} A_{mn})^2]$$

$$+ 2(1-\nu)[\alpha_m A'_{mn}(\frac{n}{a} B'_{mn} + \frac{1}{a} C'_{mn}) + \frac{1}{4}(\alpha_m B'_{mn} - \frac{n}{a} A'_{mn})^2]\}$$

$$+ \frac{\pi Eh^3 a\ell}{48(1-\nu^2)} \sum_{m=1}^{M} \sum_{n=0}^{N} [(\{\frac{1}{a^2} (1-n^2) - \alpha_m^2\}C_{mn})^2$$

$$+ \{(\frac{1}{a^2} (1-n^2) - \alpha_m^2)C'_{mn}\}^2$$

$$+ 2(1-\nu)[\alpha_m^2 C_{mn}^2 (\frac{1}{a^2} - \frac{n^2}{a^2}) + (\frac{n}{2a^2} A_{mn} + \frac{\alpha_m}{2a} B_{mn} + \frac{\alpha_{m(n)}}{a} C_{mn})^2]$$

$$+ 2(1-\nu)[\alpha_m^2 C'^2_{mn}(\frac{1}{a^2} - \frac{n^2}{a^2}) + (\frac{n}{2a^2} A'_{mn} + \frac{\alpha_m}{2a} B'_{mn} + \frac{\alpha_{m(n)}}{a} C'_{mn})^2] \quad (2.7)$$

The strain energy of a stringer at the location $\phi = \phi_r$ (see Appendix A) is:

$$V_{sr} = \frac{1}{2} \int_0^{\ell} (E_r^* A_r (\partial\zeta_r/\partial x)^2 + E_r^* I_{z'r} (\partial^2 \xi_r/\partial x^2)^2 + E_r^* I_{y'r} (\partial^2 \eta_r/\partial x^2)^2 + \quad (2.8)$$

$$2E_r^* I_{y'z'} (\partial^2 \xi_r/\partial x^2)(\partial^2 \eta_r/\partial x^2) + E_r^* C_{wr} (\partial^2 \theta_r/\partial x^2)^2 + (GJ)_r \left(\frac{\partial \theta_r}{\partial x}\right)^2 dx$$

where (ζ_r, ξ_r, η_r) denote the displacements of the shear center of the rth stringer (see Figure 2). It is assumed that the stringer is rigidly attached to the shell along a single line (attachment line) at the middle surface of the shell. Usually, the attachment line is assumed to be at the middle of the flange (see reference [5]).

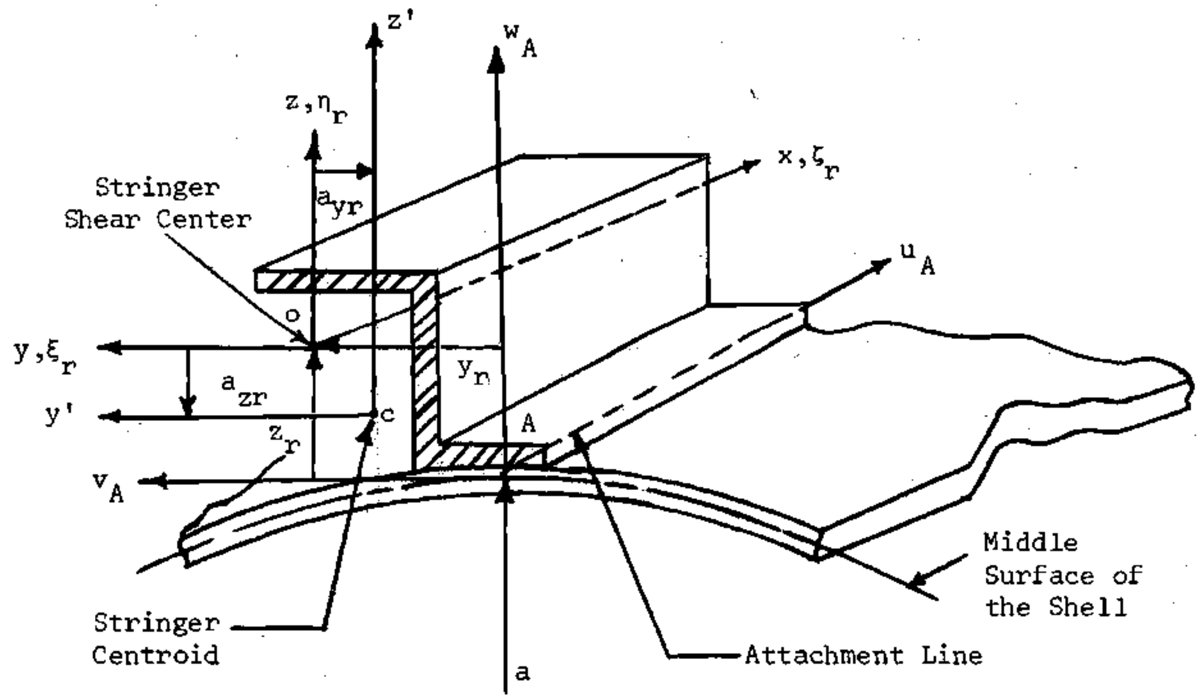


Figure 2. Stringer Coordinate Systems and Notation

Having determined an attachment line, we denote the displacements of this line by (u_A, v_A, w_A). By using conventional thin-walled beam theory the displacements of any point in the stringer (u_{sr}, v_{sr}, w_{sr}) are related to the displacements of the shear center (ζ_r, ξ_r, η_r) by the following linear transformation assuming $a_y = a_z = 0$ and no warping:

$$u_{sr} = \zeta_r - y \frac{\partial \xi_r}{\partial x} - z \frac{\partial \eta_r}{\partial x} , \tag{2.9}$$

$$v_{sr} = \xi_r - z\,\Theta_{sr},$$

$$w_{sr} = \eta_r + y\,\Theta_{sr}$$

where Θ_{sr} denotes the angle of twist of the rth stringer. Therefore, the displacements of the attachment line are related to the displacements of the coordinate origin and shear center by:

$$\zeta_r = u_A + y_r \frac{\partial \xi_r}{\partial x} + z_r \frac{\partial \eta_r}{\partial x}, \qquad (2.10)$$

$$\xi_r = v_A + z_r\,\Theta_{sr},$$

$$\eta_r = w_A - y_r\,\Theta_{sr}.$$

From the second of Equations (2.10), we have

$$\frac{\partial \xi_r}{\partial x} = \frac{\partial v_A}{\partial x} + z_r \frac{\partial \Theta_{sr}}{\partial x} \qquad (2.11)$$

and from the third of Equations (2.10), we have

$$\frac{\partial \eta_r}{\partial x} = \frac{\partial w_A}{\partial x} - y_r \frac{\partial \Theta_{sr}}{\partial x} \qquad (2.12)$$

Inserting Equations (2.11) and (2.12) into the first of equations (2.10) yields:

$$\zeta_r = u_A + y_r \frac{\partial v_A}{\partial x} + z_r \frac{\partial w_A}{\partial x} \tag{2.13}$$

and therefore the displacements of the shear center are given by: (assuming $a_y = a_z = 0$):

$$\zeta_r = u_A + y_r \frac{\partial v_A}{\partial x} + z_r \frac{\partial w_A}{\partial x},$$

$$\xi_r = v_A + z_r \Theta_{sr}, \tag{2.14}$$

$$\eta_r = w_A - y_r \Theta_{sr}.$$

The stringer angle of twist is approximated by:

$$\Theta_{sr} = \frac{1}{a} \frac{\partial w_A}{\partial \phi} \tag{2.15}$$

If u, v, and w denote the middle surface displacement of the shell, then the midsurface displacements along the attachment line are given by:

$$u_A = u(x,\phi_r,t), \tag{2.16}$$

$$v_A = v(x,\phi_r,t),$$

$$w_A = w(x,\phi_r,t).$$

Inserting Equations (2.16) and (2.15) into Equations (2.14) gives the following expressions for the displacements of the shear center:

$$\zeta_r = u + y_r \frac{\partial v}{\partial x} + z_r \frac{\partial w}{\partial x} , \tag{2.17}$$

$$\xi_r = v + \frac{z_r}{a} \frac{\partial w}{\partial \phi} ,$$

$$\eta_r = w - \frac{y_r}{a} \frac{\partial w}{\partial \phi} .$$

Substituting Equations (2.17) into Equation (2.8) yields:

$$\begin{aligned}
V_{sr} = \frac{1}{2} \int_0^{\ell} \{ & E_r^* A_r \left(\frac{\partial u}{\partial x}\right)^2 + 2E_r^* A_r y_r \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 v}{\partial x^2}\right) + 2E_r^* A_r z_r \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \\
& E_r^* A_r y_r^2 \left(\frac{\partial^2 v}{\partial x^2}\right)^2 + E_r^* A_r z_r^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + E_r^* I_{z'r} \left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \\
& E_r^* I_{y'r} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2E_r^* I_{y'z'r} \left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x^2}\right) + \frac{(GJ)_r}{a^2} \left(\frac{\partial^2 w}{\partial x \partial \phi}\right)^2 \}_{\phi=\phi_r} dx + \\
\frac{1}{2} \int_0^{\ell} \{ & + 2E_r^* I_{z'r} \left(\frac{z_r}{a}\right)\left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right) + E_r^* I_{z'r} \left(\frac{z_r}{a}\right)^2 \left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right)^2 - \\
& 2E_r^* I_{y'r} \frac{y_r}{a} \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right) + \frac{E_r^* C_{wr}}{a^2} \left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right)^2 + \\
& E_r^* I_{y'r} \left(\frac{y_r}{a}\right)^2 \left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right)^2 - 2E_r^* I_{y'z'r} \left(\frac{y_r}{a}\right)\left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right) + \\
& 2E_r^* I_{y'z'r} \left(\frac{z_r}{a}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right) - 2E_r^* I_{y'z'r} \left(\frac{y_r z_r}{a}\right)\left(\frac{\partial^3 w}{\partial x^2 \partial \phi}\right)^2 \}_{\phi=\phi_r} dx
\end{aligned} \tag{2.18}$$

where the first integral represents the strain energy due to extension, bending and torsion of the stringers (including the effect of eccentricity of the stringer) and the second integral represents the strain energy due to warping and coupling between the twisting and bending.

Substituting Equations (2.1) into (2.18) and integrating yields the following expression for the strain energy at the location $\phi = \phi_r$:

$$V_{sr} = \frac{E_r^* A_r \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[-\alpha_m A_{mi} \cos(i\phi_r) - y_r \alpha_m^2$$

$$B_{mi} \sin(i\phi_r) - z_r {}_m^2 C_{mi} \cos(i\phi_r)][-\alpha_m A_{mj}$$

$$\cos(j\phi_r) - y_r {}_m^2 B_{mj} \sin(j\phi_r) - z_r \alpha_m^2 C_{mj} \cos(j\phi_r)] +$$

$$2[-\alpha_m A_{mi} \cos(i\phi_r) - y_r {}_m^2 B_{mi} \sin(i\phi_r) - z_r \alpha_m^2 C_{mi}$$

$$\cos(i\phi_r)][-\alpha_m A'_{mj} \sin(j\phi_r) + \alpha_m^2 y_r B'_{mj} \cos(j\phi_r) -$$

$$\alpha_m^2 z_r C'_{mj} \sin(j\phi_r)] + [-\alpha_m A'_{mi} \sin(i\phi_r) + \alpha_m^2 y_r B'_{mi}$$

$$\cos(i\phi_r) - \alpha_m^2 z_r C'_{mi} \sin(i\phi_r)][-\alpha_m A'_{mj} \sin(j\phi_r) + \alpha_m^2 y_r$$

$$B'_{mj} \cos(j\phi_r) - \alpha_m^2 z_r C'_{mj} \sin(j\phi_r)] + \frac{E_r^* I_{y'r} \ell}{4}$$

$$\sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\alpha_m^2 C_{mi}][\alpha_m^2 C_{mj}] \cos(i\phi_r) \cos(j\phi_r) +$$

$$2[\alpha_m^2 C_{mi}][\alpha_m^2 C'_{mj}]\cos(i\phi_r)\sin(j\phi_r) + [\alpha_m^2 C'_{mi}]$$

$$[\alpha_m^2 C'_{mj}]\sin(i\phi_r)\sin(j\phi_r)\} +$$

$$\frac{E_r^* I_{z'r} \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[-\alpha_m^2 B_{mi} + \frac{z_r}{a}(\alpha_m^2)(i)C_{mi}]$$

$$[-\alpha_m^2 B_{mj} - \frac{z_r}{a}(\alpha_m^2)(j)C_{mj}]\sin(i\phi_r)\sin(j\phi_r) +$$

$$2[-\alpha_m^2 B_{mi} + \frac{z_r}{a}(\alpha_m^2)(i)C_{mi}][\alpha_m^2 B'_{mj} - \frac{z_r}{a}(\alpha_m^2)(j)C'_{mj}]$$

$$\sin(i\phi_r)\cos(j\phi_r) + [\alpha_m^2 B'_{mi} - C'_{mi}\left(\frac{z_r}{a}\right)(\alpha_m^2)(i)]$$

$$[\alpha_m^2 B'_{mj} - \frac{z_r}{a}(\alpha_m^2)(j)C'_{mj}]\cos(i\phi_r)\cos(j\phi_r) +$$

$$\frac{E_r^* I_{y'z'r} \ell}{2} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\alpha_m^2 C_{mi}][\alpha_m^2 B_{mj} + \frac{z_r}{a}(\alpha_m^2)(j)C_{mj}]$$

$$\cos(i\phi_r)\sin(j\phi_r) + [\alpha_m^2 C_{mi}][\alpha_m^2 B'_{mj} -$$

$$\left(\frac{z_r}{a}\right)(\alpha_m^2)(j)C'_{mj}]\cos(i\phi_r)\cos(j\phi_r) + [\alpha_m^2 C'_{mi}]$$

$$[\alpha_m^2 B_{mj} - \left(\frac{z_r}{a}\right)(\alpha_m^2)(j)C_{mj}]\sin(i\phi_r)\sin(j\phi_r) -$$

$$[\alpha_m^2 C'_{mi}][\alpha_m^2 B'_{mj} - \left(\frac{z_r}{a}\right)(\alpha_m^2)(j)C'_{mj}]\sin(i\phi_r)\cos(j\phi_r) +$$

$$\frac{(GJ)_r \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[-(i/a)\alpha_m C_{mi}][-(j/a)\alpha_m C_{mj}] +$$

$$+ 2[-(i/a)\alpha_m C_{mi}][(j/a)\alpha_m C'_{mj}]\sin(i\phi_r)\cos(j\phi_r) +$$

$$[(i/a)\alpha_m C'_{mi}][(j/a)\alpha_m C'_{mj}]\cos(i\phi_r)\cos(j\phi_r) +$$

$$\frac{E_r^* C_{wr} \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\alpha_m^2(i)C_{mi}\sin(i\phi_r)][\alpha_m^2(j)$$

$$C_{mj}\sin(j\phi_r)] - 2[\alpha_m^2(i)C_{mi}\sin(i\phi_r)]$$

$$[\alpha_m^2(j)C'_{mj}\cos(j\phi_r)] + [\alpha_m^2(i)C'_{mi}\cos(i\phi_r)]$$

$$[\alpha_m^2(j)C'_{mj}\cos(j\phi_r)] \tag{2.19}$$

Therefore, the total strain energy of the system is:

$$V = V_{SHELL} + \sum_{r=1}^{K} V_{sr} \tag{2.20}$$

where K is the total number of stringers attached to the shell's surface.

Kinetic Energy of System

The kinetic energy of a cylindrical shell (including rotatory inertia) is given by Ojalvo and Newman [5] as the following:

$$T_{SHELL} = \frac{\rho h a}{2} \int_0^{\ell} \int_0^{2\pi} \{\dot{u}^2 + \dot{v}^2 + \dot{w}^2 +$$

$$+ \beta a^2 \left[\left(\frac{\partial \dot{w}}{\partial x}\right)^2 + \frac{1}{a^2}\left(\frac{\partial \dot{w}}{\partial \phi} - \dot{v}\right)^2\right]\} \, dx d\phi \qquad (2.21)$$

where ρ is the mass density of the shell, β is the shell bending-stiffness parameter, and the dot indicates differentiation with respect to time. Inserting the expressions for the assumed displacements into Equation (2.21) and integrating gives:

$$T_{SHELL} = \frac{\pi \rho h a \ell}{4} \sum_{m=1}^{M} \sum_{n=0}^{N} \{(\dot{A}_{mn})^2 + (\dot{A}'_{mn})^2 + (\dot{B}_{mn})^2(1 + \beta) +$$

$$2\beta(n)\dot{C}_{mn}\dot{B}_{mn} + (\dot{B}'_{mn})^2(1+\beta) + 2\beta(n)\dot{C}'_{mn}\dot{B}'_{mn} + (\dot{C}_{mn})^2$$

$$[1 + \beta(a^2\alpha_m^2+n^2)] + (\dot{C}'_{mn})^2[1 + \beta(a^2\alpha_m^2 + n^2)]\} \qquad (2.22)$$

The kinetic energy of a typical stringer (including rotatory inertia) in terms of the middle surface displacements of the shell is given by (see Appendix A):

$$T_{sr} = \frac{\rho_r A_r}{2} \int_0^{\ell} \{\left(\dot{u} + y_r \frac{\partial \dot{v}}{\partial x} + z_r \frac{\partial \dot{w}}{\partial x}\right)^2 + \left(\dot{v} + \frac{z_r}{a}\frac{\partial \dot{w}}{\partial \phi}\right)^2 + \left(\dot{w} - \frac{y_r}{a}\frac{\partial \dot{w}}{\partial \phi}\right)^2 +$$

$$\frac{I_{pr}}{A_r}\left(\frac{1}{a}\frac{\partial \dot{w}}{\partial \phi}\right)^2 - \frac{2a_{yr}}{a}(\dot{w})\left(\frac{\partial \dot{w}}{\partial \phi}\right) + 2\frac{a_{zr}}{a}(\dot{v})\left(\frac{\partial \dot{w}}{\partial \phi}\right)\}_{\phi=\phi_r} dx +$$

$$\frac{\rho_r I_{y'r}}{2} \int_0^{\ell} \left(\frac{\partial \dot{w}}{\partial x}\right)^2_{\phi=\phi_r} dx + \frac{\rho_r I_{z'r}}{2} \int_0^{\ell} \left(\frac{\partial \dot{v}}{\partial x}\right)^2_{\phi=\phi_r} +$$

$$\rho I_{y'z'r} \int_0^{\ell} \left(\frac{\partial \dot{v}}{\partial x}\right)\left(\frac{\partial \dot{w}}{\partial x}\right) dx \qquad (2.23)$$

Inserting Equations (2.1) into (2.23) and performing the integrations yield:

$$T_{sr} = \frac{\rho_r A_r \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\dot{A}_{mi}\cos(i\phi_r) + \alpha_m y_r \dot{B}_{mi}$$

$$\sin(i\phi_r) + \alpha_m z_r \dot{C}_{mi}\cos(i\phi_r)][\dot{A}_{mj}\cos(j\phi_r) +$$

$$\alpha_m y_r \dot{B}_{mj}\sin(j\phi_r) + \alpha_m z_r \dot{C}_{mj}\cos(j\phi_r)] +$$

$$[\dot{A}'_{mi}\sin(i\phi_r) + \alpha_m y_r \dot{B}'_{mi}\cos(i\phi_r) + \alpha_m z_r \dot{C}'_{mi}$$

$$\sin(i\phi_r)][\dot{A}'_{mj}\sin(j\phi_r) - \alpha_m y_r \dot{B}'_{mj}\cos(j\phi_r) + \alpha_m z_r$$

$$\dot{C}'_{mj}\sin(j\phi_r)] + 2[\dot{A}_{mi}\cos(i\phi_r) + \alpha_m y_r \dot{B}_{mi}\sin(i\phi_r) +$$

$$\alpha_m z_r \dot{C}_{mi}\cos(i\phi_r)][\dot{A}'_{mj}\sin(j\phi_r) - \alpha_m y_r \dot{B}'_{mj}\cos(j\phi_r)$$

$$\alpha_m z_r \dot{C}'_{mj}\sin(j\phi_r)]\} + \frac{\rho_r A_r \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[$$

$$\dot{B}_{mi} - \left(\frac{z_r}{a}\right)(i)\dot{C}_{mi}][\dot{B}_{mj} - \left(\frac{z_r}{a}\right)(j)\dot{C}_{mj}]\sin(i\phi_r)\sin(j\phi_r) +$$

$$2[\dot{B}_{mi} - \left(\frac{z_r}{a}\right)(i)\dot{C}_{mi}][-\dot{B}'_{mj} + \left(\frac{z_r}{a}\right)(j)\dot{C}'_{mj}]\sin(i\phi_r)\cos(j\phi_r)$$

$$[-\dot{B}'_{mi} + \left(\frac{z_r}{a}\right)(i)\dot{C}'_{mi}][-\dot{B}'_{mj} + \left(\frac{z_r}{a}\right)(j)\dot{C}'_{mj}]\cos(i\phi_r)\cos(j\phi_r)\}$$

$$+ \frac{\rho_r A_r \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{\dot{C}_{mi}\cos(i\phi_r) + \frac{y_r(i)}{a}\dot{C}_{mi}\sin(i\phi_r)]$$

$$[\dot{C}_{mj}\cos(j\phi_r) + \frac{y_r(j)}{a}\dot{C}_{mj}\sin(j\phi_r)] + [\dot{C}'_{mi}\sin(i\phi_r) -$$

$$\frac{y_r(i)}{a}\dot{C}'_{mi}\cos(i\phi_r)][\dot{C}'_{mj}\sin(j\phi_r) - \frac{y_r(j)}{a}\dot{C}'_{mj}\cos(j\phi_r)] +$$

$$2[\dot{C}_{mi}\cos(i\phi_r) + \frac{y_r(i)}{a}\dot{C}_{mi}\sin(i\phi_r)][\dot{C}'_{mj}\sin(j\phi_r) -$$

$$\frac{y_r(j)}{a}\dot{C}'_{mj}\cos(j\phi_r)]\} + \frac{\rho I_{pr}\ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[$$

$$(-i/a)\dot{C}_{mi}][(-j/a)\dot{C}_{mj}]\sin(i\phi_r)\sin(j\phi_r) +$$

$$2[(-i/a)C_{mi}][(j/a)\dot{C}'_{mj}]\sin(i\phi_r)\cos(j\phi_r) +$$

$$[(i/a)\dot{C}'_{mi}][(j/a)\dot{C}'_{mj}]\cos(i\phi_r)\cos(j\phi_r)\} -$$

$$\frac{\rho_r a_{yr} A_r \ell}{2a} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\dot{C}_{mi}\cos(i\phi_r)][-j\dot{C}_{mj}$$

$$\sin(j\phi_r)] + [\dot{C}_{mi}\cos(i\phi_r)][j\dot{C}'_{mj}\cos(j\phi_r)] +$$

$$[\dot{C}'_{mi}\sin(i\phi_r)][-j\dot{C}_{mj}\sin(j\phi_r)] + [\dot{C}'_{mi}\sin(i\phi_r)]$$

$$[j\dot{C}'_{mj}\cos(j\phi_r)]\} + \frac{\rho_r a_{zr} A_r \ell}{2a} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[$$

$$\dot{B}_{mi}\sin(i\phi_r)][-j\dot{C}_{mj}\sin(j\phi_r)] + [\dot{B}'_{mi}\cos(i\phi_r)][j\dot{C}_{mj}$$

$$\sin(j\phi_r)] + [\dot{B}_{mi}\sin(i\phi_r)][j'\dot{C}'_{mj}\cos(j\phi_r)] -$$

$$[\dot{B}'_{mi}\cos(i\phi_r)][j\dot{C}'_{mj}\cos(j\phi_r)]\} + \frac{\rho_r I_{y'r} \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[$$

$$\dot{C}_{mi}\alpha_m\cos(i\phi_r)][\dot{C}_{mj}\alpha_m\cos(j\phi_r)] + 2[\dot{C}_{mi}(\alpha_m)\cos(i\phi_r)]$$

$$[\dot{C}'_{mj}\alpha_m\sin(j\phi_r)] + [\dot{C}'_{mi}(\alpha_m)\sin(i\phi_r)][\dot{C}'_{mj}(\alpha_m)\sin(j\phi_r)]\}$$

$$+ \frac{\rho_r I_{z'r} \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\dot{B}_{mi}\alpha_m\sin(i\phi_r)][\dot{B}_{mj}(\alpha_m)\sin(j\phi_r)] -$$

$$2[\dot{B}_{mi}(\alpha_m)\sin(i\phi_r)][-\dot{B}'_{mj}(\alpha_m)\cos(j\phi_r)] + [\dot{B}'_{mi}(\alpha_m)$$

$$\cos(i\phi_r)][\dot{B}'_{mj}(\alpha_m)\cos(j\phi_2)]\} + \frac{\rho_r I_{y'z'r}\ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[$$

$$\dot{B}_{mi}(\alpha_m)\sin(i\phi_r)][\dot{C}_{mj}(\alpha_m)\cos(j\phi_r)] - [\dot{B}'_{mi}(\alpha_m)\cos(j\phi_r)]$$

$$[C_{mj}(\alpha_m)\cos(j\phi_r)] + [\dot{B}_{mi}(\alpha_m)\sin(i\phi_r)][\dot{C}'_{mj}(\alpha_m)$$

$$\sin(j\phi_r)] - [\dot{B}_{mi}(\alpha_m)\cos(i\phi_r)][\dot{C}'_{mj}(\alpha_m)\sin(j\phi_r)]\} \tag{2.24}$$

Therefore, the total kinetic energy of the system is:

$$T = T_{SHELL} + \sum_{r=1}^{K} T_{sr} \tag{2.25}$$

CHAPTER III

FREE VIBRATIONS OF SIMPLY SUPPORTED STIFFENED CYLINDER ACCORDING TO RITZ METHOD

The equations of motion for free vibration are derived from Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{mn}}\right) - \frac{\partial T}{\partial q_{mn}} + \frac{\partial V}{\partial q_{mn}} = 0 \tag{3.1}$$

where T and V are the kinetic and potential energy of the system, respectively and q_{mn} is A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn} or C'_{mn}. For free vibrations the generalized coordinates are harmonic functions of time; that is, we have:

$$\begin{pmatrix} A_{mn} \\ A'_{mn} \\ B_{mn} \\ B'_{mn} \\ C_{mn} \\ C'_{mn} \end{pmatrix} = \begin{pmatrix} a_{mn} \\ a'_{mn} \\ b_{mn} \\ b'_{mn} \\ c_{mn} \\ c'_{mn} \end{pmatrix} \cos \omega t \tag{3.2}$$

where ω represents the natural frequency of the composite shell. If one introduces the following notation

$$T = \omega^2 T^* \cos^2 \omega t$$

$$V = V^* e^{2i\omega t}, \text{ (free vibration)}$$

then (3.1) can be written in the characteristic form of the Ritz method:

$$\omega^2 \frac{\partial T^*}{\partial q_{mn}} - \frac{\partial V^*}{\partial q_{mn}} = 0 \tag{3.3}$$

where now q_{mn} is a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} or c'_{mn}. The expression for T^* is obtained from Equation (2.25) by replacing $\dot{A}_{mn}$, $\dot{A}'_{mn}$, $\dot{B}_{mn}$, $\dot{B}'_{mn}$, $\dot{C}_{mn}$ or $\dot{C}'_{mn}$ with a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} or c'_{mn}, respectively. Similarly, the expression for V^* is obtained from Equation (2.20) by replacing A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn} or C'_{mn} with a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} or c'_{mn}.

Inserting expressions for the derivatives of T^* and V^* into Equation (3.3), a set of 6N equations for each axial mode (wave number m) is generated because the axial modes are not coupled for the simply supported case. These equations are linear in the amplitudes of the displacement series and form an eigenvalue problem. In matrix form, these equations are:

$$[K]\{q\} - \Omega\,[M]\{q\} = 0 \tag{3.4}$$

where

[K] is the stiffness matrix, 6Nx6N,

[M] is the mass matrix, 6Nx6N,

{q} is the matrix of generalized coordinates,

and

$\Omega = (1-\nu^2)\rho a^2\omega^2/E$ is the frequency parameter.

Equations (3.4) can be reduced to the standard form $[C]\{Y\} = \lambda\{Y\}$, by factoring [K] into the product of a lower triangular matrix [L] times its transpose $[L]^T$, as follows:

$$[K] = [L][L]^T \tag{3.5}$$

this is permissible since [K] is positive definite and symmetric. Pre-multiplying (3.4) by $[L]^{-1}$ yields:

$$[L]^{-1}[M]\{q\} = \lambda[L]^{-1}[L][L]^T\{q\}, \tag{3.6}$$

or

$$[L]^{-1}[M]\{q\} = \lambda[L]^T\{q\} \tag{3.7}$$

where

$$\lambda = 1/\Omega.$$

Rewriting this equation as

$$[L]^{-1}[M]\,\{[L]^{-1}\}^T\,[L]^T\{q\} = \lambda[L]^T\{q\} \tag{3.8}$$

since

$$\{[L]^{-1}\}^T[L]^T = [I]$$

and letting

$$[C] = [L]^{-1}[M]\{[L]^{-1}\}^T \quad \text{and} \quad \{Y\} = [L]^T\{q\}$$

we obtain the standard form

$$[C]\{Y\} = \lambda\{Y\} \tag{3.9}$$

where [C] is a symmetric 6Nx6N matrix.

The stiffness matrix [K] is given by its submatrices as:

$$[K] = \left[\begin{array}{c:c:c:c:c:c} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ \hdashline (k_{12})^T & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ \hdashline (k_{13})^T & (k_{23})^T & k_{33} & k_{34} & k_{35} & k_{36} \\ \hdashline (k_{14})^T & (k_{24})^T & (k_{34})^T & (k^a_{11}) & (k^a_{12}) & (k^a_{13}) \\ \hdashline (k_{15})^T & (k_{25})^T & (k_{35})^T & (k^a_{12})^T & k^a_{22} & k^a_{23} \\ \hdashline (k_{16})^T & (k_{26})^T & (k_{36})^T & (k^a_{13})^T & (k^a_{23})^T & (k^a_{33}) \end{array}\right] \tag{3.10}$$

The mass matrix [M] is given in terms of its submatrices as:

$$[M] = \left[\begin{array}{c:c:c:c:c:c} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ \hdashline (m_{12})^T & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ \hdashline (m_{13})^T & (m_{23})^T & m_{33} & m_{34} & m_{35} & m_{36} \\ \hdashline (m_{14})^T & (m_{24})^T & (m_{34})^T & m_{11}^a & m_{12}^a & m_{13}^a \\ \hdashline (m_{15})^T & (m_{25})^T & (m_{35})^T & (m_{12}^a)^T & m_{22}^a & (m_{23}^a) \\ \hdashline (m_{16})^T & (m_{26})^T & (m_{36})^T & (m_{13}^a)^T & (m_{23}^a)^T & (m_{33}^a) \end{array}\right] \qquad (3.11)$$

The elements in the matrices [K] and [M] are given in Appendix C. The column matrix {q} of generalized coordinates is written as:

$$\{q\} = \left\{\begin{array}{c} \bar{a} \\ \hdashline \bar{b} \\ \hdashline \bar{c} \\ \hdashline \bar{a}' \\ \hdashline \bar{b}' \\ \hdashline \bar{c}' \end{array}\right\} \qquad (3.12)$$

where the elements $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{a}'$, $\bar{b}'$, and $\bar{c}'$ are column vectors whose components are given by:

$$\bar{a}_j = a_{mj} \qquad (3.13)$$

$$\bar{b}_j = b_{mj}$$

$$\bar{c}_j = c_{mj}$$

$$\vdots$$

$$\bar{c}'_j = c'_{mj}$$

where m is a fixed axial mode number and $j = 1,2,\ldots,N$.

The addition of stringers on a cylindrical shell has a significant effect on the circumferential modes of the structure. The circumferential modes with different wave numbers n_1 and n_2 are coupled and the symmetric modes (a_{mn},b_{mn},c_{mn}) are also coupled to the antisymmetric modes $(a'_{mn},b'_{mn},c'_{mn})$. However, it is possible to simplify Equations (3.4) by considering certain types of stringer configurations.

If the stringers have an axis of symmetry $(I_{yzi}=0)$ and are identical and equally spaced around the circumference of the shell then it is possible to uncouple the symmetric and antisymmetric modes. If the stringers are equally spaced, then the stringer distribution is given by:

$$\phi_j = \frac{2\pi j}{K} \qquad j = 1,2,\ldots,k$$

where K is the total number of stringers. This distribution is symmetric with respect to the x-z plane. For equally spaced, identical stringers, the terms which couple the circumferential modes of different wave numbers contain the following sums:

$$X = \sum_{j=1}^{K} \sin\left(\frac{2\pi j}{K} n_1\right) \sin\left(\frac{2\pi j}{K} n_2\right) \tag{3.14}$$

$$Z = \sum_{j=1}^{K} \sin\left(\frac{2\pi j}{K} n_1\right) \cos\left(\frac{2\pi j}{K} n_2\right)$$

Miller [2] showed that these sums are given by:

$$X = \begin{pmatrix} 0 \quad \text{if } n_1+n_2 \neq 0, \pm k, \pm 2k, \ldots \\ k/2 \text{ if } (n_1+n_2) \text{ or } (n_1-n_2)=0, \pm k, \pm 2k, \ldots \\ 0 \quad \text{if } (n_1+n_2) \text{ and } (n_1-n_2)=0, \pm k, \pm 2k, \ldots \end{pmatrix}$$

$$Y = \begin{pmatrix} 0, n_1 \pm n_2 \neq 0, \pm k, \pm 2k, \ldots \\ k/2, (n_1+n_2) \text{ or } (n_1-n_2)=0, \pm k, \pm 2k, \ldots \\ k, (n_1+n_2) \text{ and } (n_1-n_2)=0, \pm k, \pm 2k, \ldots \end{pmatrix} \tag{3.15}$$

$$Z = \left(0 \text{ for all } n_1 \text{ and } n_2\right)$$

It may be noted that X=Y=0 if $n_1=n_2$ and $n_1+n_2<K$ which states that there is no coupling between circumferential modes with wave numbers n_1 and n_2 if $n_1 \neq n_2$ and $n_1+n_2<K$. Therefore, the larger the number of stringers, the less number of coupled modes for the same number of terms in an assumed displacement series. However, if one has a few stringers, say 8, then there is no coupling between the first circumferential mode and modes 1, 2, 3, 4, 5, 6 but the first and seventh modes are coupled. Miller [2] and Ojalvo and Newman [5] used

the fact that if $K \gg (n_1+n_2)$ (i.e., if the stiffener spacing is small compared to the distance between successive circumferential nodal lines) then it is permissible to drop the sum on n in Equations (2.1) and assume that:

$$u(x,\phi) = (A_{mn}\cos n\phi + A'_{mn}\sin n\phi)\cos \frac{m\pi x}{\ell}$$

$$v(x,\phi) = (B_{mn}\sin n\phi - B'_{mn}\cos n\phi)\sin \frac{m\pi x}{\ell} \qquad (3.16)$$

$$w(x,\phi) = (C_{mn}\cos n\phi + C'_{mn}\sin n\phi)\sin \frac{m\pi x}{\ell}$$

Thus, the behavior of a shell with many stringers is that of a uniform orthotropic cylinder where the circumferential mode is a pure harmonic. This assumption is in very good agreement with experimental results for shells with a large number of stringers.

In the literature cited, one finds different expressions for the strain and kinetic energies of a shell and stringers. For example, Miller [2] considers only the strain energy of the stringers arising from the terms $\frac{EA}{2}\int_0^\ell \left(\frac{\partial \mu_o}{\partial x}\right)^2 dx$ and $\frac{EI_{zz}}{2}\int_0^\ell \left(\frac{\partial^2 w_o}{\partial x^2}\right)^2 dx$ as important but Ojalvo and Newman [5] include all the strain energy terms but warping and also neglect rotatory inertia of the stringers in the kinetic energy expression. Egle and Sewall [6] assume the shear center coincides with the centroid of the stringers and neglect the I_y and I_{yz} terms in the strain energy of the stringers. However, they account for transverse stiffening effects by incorporating "stiffening ratios" in the bending energy expression for the shell which then takes the form:

$$\frac{D}{2}\int_0^{2\pi}\int_0^{\ell}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{D_y}{D}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \nu\left(1+\frac{D_y}{D}\right)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] a\,dx\,d\phi$$

As a result of this approximation, the fact that $I_{yzi} = 0$ in order to uncouple the symmetric and antisymmetric modes does not explicitly come up. Further, it should be noted that E_r^* in the present theory is $E_s/(1-\nu^2)$ and not Young's modulus of the thin-walled beam as is used in other references. This is a consequence of the assumption that in Vlasov's thin-walled beam theory one assumes that the section contour is inextensible which yields the result that the relation between longitudinal extensions ε_x and normal stresses σ_x is

$$\sigma_x = \frac{E_s}{1 - \nu^2}\,\varepsilon_x \tag{3.17}$$

and not

$$\sigma_x = E_s \varepsilon_x \tag{3.18}$$

as in elementary beam theory. However, if one makes the assumption that the quantity ν^2 is negligible compared with unity as was done by Vlasov [10] then we get the usual stress-strain relation of beam theory that:

$$\sigma_x = E_s \varepsilon_x .$$

Numerical Results and Discussion

Computer programs were written to solve the eigenvalue problem (i.e., Equation 3.4) in the ALGOL for use with the Burrough's B5500 digital computer. Because of the relatively large size of the eigenvalue problem, the equally spaced stringer distribution was assumed. For symmetric stringers this enables one to calculate the symmetric and antisymmetric modes separately. The eigenvalues and eigenvectors of (3.4) were calculated by the Householder-Givens method (see references [11] through [14]). This is a very fast technique which was found to be much faster and as accurate as the well-known Jacobi method. Even so, the subroutines employed required 30 minutes for the solution of a typical (90x90) size problem.

Free Vibration

Table 2 and Table 5 compare the frequencies of a simply supported cylindrical shell with four and eight internal stringers, respectively, as calculated by the present analysis with the experimental results of Schnell [4]. The material properties and geometry are given by case 1 in Table 1. The eigenvalues are compared for a fixed value of the axial mode number with a number of circumferential modes associated with it. Since the circumferential modes are coupled and therefore not harmonic as for the case of the unstiffened shell, there is a problem in identifying these modes. Usually, the term in the series with the largest coefficient is the predominant term in the mode shape. If this is the nth term in the series then the mode will be called the nth circumferential mode. Thus, n=3 refers to a wave form around the circumference with three waves but the waves are simply not harmonic. From Table 2, it is seen that there are two natural frequencies, one is associated with a symmetric mode and the other is associated with an antisymmetric mode. These frequencies are not generally the same.

The symmetric and antisymmetric frequencies for the shell with four stringers are the same for odd values of n, however. The reason for this is that for the odd symmetric modes, the stringers at $\phi=0,\pi$ are at antinodal lines and the ones at $\phi=\pi/2$, $3\pi/2$ are at nodes. For the case of the odd antisymmetric modes the stringers at $\phi=0,\pi$ are at node lines, but the ones at $\phi=\pi/2$, $3\pi/2$ are at antinodal lines. Therefore, the two circumferential modes are the same and one obtains the same frequencies. However, for the even symmetric modes all of the stringers are in bending (see Figure 5(a) for the symmetric modes), whereas for the even antisymmetric modes all of the stringers are in torsion (see Figure 5(b) for the antisymmetric modes). Since the strain energy of the stringers in bending is not generally the same as for torsion, the frequencies for the even symmetric modes will not be the same as the frequencies for the antisymmetric modes. Good agreement is noted for the theoretical values as compared with the experimental results. It may be noted that the experimental results give only a single frequency for a given circumferential mode. The reason for this is not known but it appears that a symmetric mode may have been excited in some cases (e.g., m=1, n=8) and an antisymmetric mode in others (e.g., m=1, n=12). The case for eight stringers is analogous. The symmetric and antisymmetric frequencies are the same for the odd values of n. The frequencies for the symmetric and antisymmetric modes are the same for n=2,6,10 since four stringers are at nodal lines and four stringers are at antinodal lines.

Figures 3-4 illustrate the fact that the frequency curves for the stiffened shell are analogous to those of the unstiffened shell (see reference [15]). For a given axial mode the frequencies decrease with increasing n down to a minimum and then increase again. The results also indicate that the value of n at which the minimum frequency occurs depends upon the axial wavelength. As m increases, the value of n corresponding to the minimum frequency also increases or remains the same. The eigenvalues were calculated using Donnell's shell theory and neglecting rotatory inertia of the shell wall. It was also found for the hat-shaped stringer ($\sqcap$) that the frequencies and mode shapes were unchanged if one mode the further assumption that the shear center coincides with the centroid (i.e., $a_{yr} = a_{zr} = 0$). Since $a_{yr} = 0$ for this type of stringer, this result could be expected since the a_{zr} term contributes only a negligible amount of inertia to the structure and one has already accounted for the primary effects of such stringer eccentricity effects in the z_r terms.

Tables 3-4 give the relative magnitudes of the components of some selected eigenvectors for the radial frequencies. These tables show the effect of circumferential modal coupling on the structure for four stringers. In the tables, the components a_{mn},b_{mn} and c_{mn} are normalized relative to the predominant radial component. Consider, for instance, the mode m=2, n=5 in Table 4. The dominant radial amplitude is c_{25}. All of the components have been normalized with respect to c_{25}. If the shell were not stiffened then c_{25} would be the only non-zero component in this particular mode. The effect of attaching longitudinal ribs has been to couple these circumferential modes with different wave numbers together. It may be noted that the coefficients in the mode m=2, n=1 are not decreasing as rapidly as for the other modes. This is reflected in the mode shapes (see Figure 5(b)) in which the mode (m=2, n=1) is not as smooth as the other mode shapes. The absolute value of the ratio of the largest secondary component to the dominant component is a measure of the strength of the coupling. In the mode (m=2, n=5), this ratio is the largest for coupling between the ninth and fifth modes (ratio=.06). However, for the mode (m=2,n=7), the ratio is the largest for coupling between the seventh and fifth modes (ratio=.13). In Table 3, considering the mode (m=4, n=1), the dominant amplitudes for the complete set of eigenvectors are a_{41}, b_{41}, and c_{41}, but for the mode (m=4, n=2) the dominant amplitudes are a_{42}, b_{42}, and c_{42}.

In Figures 5 and 6 some selected theoretical mode shapes are presented for the case of four internal stringers. These figures show the normalized radial deflection (w) plotted against a nondimensional circumferential coordinate. In addition Figure 5 also shows the normalized longitudinal deflection (u) and circumferential deflection (v) for the location $x = \ell/4$. The positions of the stringers on the shell are also given.

The eigenvalues for 4 axial modes and 25 symmetric circumferential modes are given in Tables 6-9 for the case of the internally stiffened cylinders with four or eight equally spaced stringers. It may be observed from these tables that for the radial frequencies, the frequencies of the shell reaches higher values with m increasing than with n increasing. This is because the contribution of the stringer to the potential energy increases with m while its kinetic energy is independent of m and n. Thus, one has a stiffer system with higher frequencies. Arnold and Warburton [15] showed that for the unstiffened shell, that the displacement associated with the lowest frequency is predominantly radial. The intermediate frequency corresponds to mainly a longitudinal motion for small values of m and the circumferential component is associated with the highest frequencies. The intermediate and highest frequencies are associated almost entirely with extensional vibrations, in which the strain energy is large relative to that of the lowest frequencies. These characteristics are also true of the stringer stiffened shell as may be seen from examination of Tables 6-9. Usually the radial frequencies are the ones that are most easily excited and are the ones which are observed experimentally.

Yu [16] used the dynamic counterpart of Donnell's equations to investigate the free vibration characteristics of an unstiffened cylindrical shell. He obtained results in good agreement with Flügge's results for the more exact shell theory. Smith [17] presented a method for obtaining a closed form solution to Donnell's equations without first making any simplifications. He obtained good agreement between experimental and theoretical results for simply supported and clamped-clamped shells. Tables 10-13 show a comparison of Flügge's shell theory versus Donnell's shell theory for the stiffened cylindrical shell treating the stringers as discrete elements and neglecting rotatory inertia of the shell. These results are in very good agreement for the numerical example considered.

In Table 14, the results of Donnell's shell theory are compared with the orthotropic results presented in reference [4] for the case of a cylindrical shell stiffened by means of four internal stringers. It may be immediately noticed that no results are given for n=1 for the orthotropic case. This is because the orthotropic theory yields an equivalent shell with too great a thickness and hence the shell is much stiffer than the actual stiffened cylinder. As a result, the frequencies are too high and no meaning can be attached to the results. It may be further noticed that the integer n for the orthotropic case corresponds to a harmonic wave. Therefore, the symmetric and antisymmetric frequencies are the same and only one frequency is given. This is, of course, in disagreement with the results of the Ritz energy method. The comparison of results shows reasonable agreement for most modes. However, there are some considerable discrepancies. For example, the antisymmetric frequency in the mode m=4, n=4 differs from the orthotropic result by about 23 per cent. Schnell and Heinrichsbauer [4] did not present any results stating as to when it was possible to use the orthotropic theory for a shell with only a few stringers to predict frequencies. This does not in general seem feasible because of the large parameters involved.

Table 15 compares the theoretical frequencies for four internal stringers (⎍) by neglecting rotatory inertia of the stringers and assuming the shear center coincides with the centroid (a_z=0). The assumption $a_z = 0$ had no significant effect on the frequencies of the stiffened shell (less than 1/10 of 1 per cent change) and also yielded the same mode shapes. The results of these tables show excellent agreement. The primary rotatory inertia term was the contribution from the I_{zz} term. This comes from the m_{22} sub-matrix (see Appendix C, Equation C-22). Hence, if $\alpha_m^2 I_{zz}/A_i << 1$, then this term has a negligible effect. The term becomes more important at high axial wave numbers, however.

The effect of neglecting rotatory inertia on the theoretical longitudinal and circumferential frequencies for four internal stringers is noted in Tables 16 and 17. A close examination of these tables reveals that making such an assumption also has little effect on the natural frequencies. Since the strain energy of the shell becomes very high for these modes, it seems reasonable to expect this result, although no experimental data is available.

A comparison of the results of the present theory retaining in-plane inertia versus the results of reference [6] which employed stiffening ratio's and neglected in-plane inertias for a shell stiffened by 60 stringers is presented in Table 18. The geometric properties are given by case 3 in Table 1. For this stringer configuration mode shapes are almost pure harmonics. Serious disagreement can be noted for the results shown with the discrepancy increasing for higher circumferential waves. For example, in the mode m=2, n=15 the difference is about 22 per cent. The results for the analysis of reference [6] are higher since the in-plane inertias were neglected.

The assumption of neglecting rotatory inertia of the stringers for eight stringers is compared with the experimental results in Tables 19 and 20. Again, excellent agreement is noted for 4 axial waves and 25 circumferential waves (less than 2 per cent difference in most cases).

Table 21 compares the results of the present theory with the experimental results of Hoppman [1] for a cylindrical shell with 16 external stringers (type ▨) which are equally spaced around the circumference. The stringer properties are given by case 2 in Table 1. The orthotropic results are also compared. The results of the present analysis are given for the symmetric modes.

It may be noted that no orthotropic calculations are made for the n=1 mode since the "smear" analysis predicts frequencies which are too high. The results of the present analysis using Donnell's shell theory and retaining in-plane inertia terms are in very good agreement with the orthotropic results for the n≥3 modes. The worse agreement comes for the n=2 mode (i.e., m=4, n=2) where the results differ about 20 per cent. The orthotropic theory predicts frequencies higher than the present analysis in this mode. One reason for this is that the orthotropic theory probably overstiffens the shell for this case, also.

The results between the present theory and the experiments are in good agreement except for the case when m=4 where the experimental results are higher than the results of the present analysis. This trend may be noted in the other modes also (i.e., m=3, n=4,5 and m=2, n=4,5). One possible reason for this discrepancy is that the boundary conditions may not have been exactly duplicated. Perhaps a longitudinal and/or rotational restraint may have been introduced.

The assumption of neglecting "rotatory" inertia of the stringers was investigated for this case, also. It was found that the difference between theoretical frequencies by employing this assumption was less than 1 per cent. The assumption of neglecting in-plane inertias of the shell was also investigated. It was found that this assumption caused much higher theoretical frequencies in the higher axial modes ($m \geq 2$) because of the decrease in the inertia of the structure. It is concluded that the assumptions of neglecting in-plane inertias of the shell is not valid and it is recommended that this assumption not be employed.

Table 1. Properties of Stringer Stiffened Shells

Case	1*	2	3
a, in	7.657	1.925	9.55
h, in	1.826×10^{-2}	6.50×10^{-2}	2.80×10^{-2}
E_c, psi	24.8×10^{6}	10^{7}	10.5×10^{6}
ρ_c, ρ_s, lb-sec^2/in^4	6.95×10^{-4}	2.59×10^{-4}	2.56×10^{-4}
ν	0.30	0.30	0.30
ℓ, in	38.85	15.53	23.75
$(GJ)_s$, lb-in^2	10.76	1482	360
A_s, in^2	1.589×10^{-2}	2.625×10^{-2}	2.899×10^{-2}
Z_s, in	-.208	+.137	+.152
a_y, in	0.0	0.0	0.0
a_z, in	.372	0.0	0.0
I_{zz}, in^4	1.493×10^{-4}	3.42×10^{-5}	2.227×10^{-5}
I_{yy}, in^4	3.364×10^{-4}	9.651×10^{-5}	2.203×10^{-3}
I_{yz}, in^4	0.0	0.0	0.0
y_s, in	0.0	0.0	0.0
E_s, psi	30×10^{6}	107	10.5×10^{6}
C_{ws}, in^6	2.96×10^{-6}	0.0	0.0
h/a	2.38×10^{-3}	3.38×10^{-2}	2.94×10^{-2}
ℓ/a	5.07	8.07	2.48

*These data together with frequency tables were received in private communication from Professor F. J. Heinrichsbauer, Institut Für Festigkeit, West Germany, on April 10, 1967. The data for case (2) was obtained from reference [1] and for case (3) from reference [6].

Table 2. Comparison of Theoretical and Experimental Radial Frequencies for Four Equally Spaced Stringers

m=1	f in cps				m=2	f in cps			
n	a	b	c	d	n	a	b	c	d
1	745	745	*	*	1	1829	1829	*	*
2	317	302	*	*	2	984	959	*	*
3	157	157	170	156	3	546	546	*	*
4	105	99	104	100	4	356	337	358	336
5	91	91	92	89	5	243	243	253	229
6	106	109	108	104	6	204	189	200	194
7	137	137	140	137	7	187	187	188	187
8	173	181	181	174	8	207	204	208	207
9	222	222	224	224	9	238	238	242	238
10	265	279	282	265	10	281	286	291	285
11	330	330	340	336	11	337	337	346	336
12	386	398	405	396	12	395	401	404	401
m=3					m=4				
1	2648	2648	*	*	1	3136	3136	*	*
2	1637	1647	*	*	2	2188	2188	*	*
3	1039	1039	*	*	3	1519	1519	*	*
4	721	673	*	*	4	1072	816	*	*
5	491	491	460	*	5	781	781	*	*
6	396	355	380	372	6	639	592	*	*
7	313	313	309	305	7	488	488	495	482
8	302	273	282	*	8	424	391	415	420
9	288	288	297	296	9	384	384	380	385
10	321	315	322	329	10	402	371	387	392
11	362	362	369	372	11	412	412	415	425
12	418	418	422	421	12	493	455	458	509

Key

a = Symmetric modes, stiffened (Donnell's theory).
b = Antisymmetric modes, stiffened (Donnell's Theory).
c = Unstiffened, experimental results, Schnell and Heinrichsbauer [6].
d = Stiffened, experimental results, Schnell and Heinrichsbauer.

Type of Stringer: .

* = No experimental results for this case.

Table 3. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Symmetric)

m=4 n=	1	2	3	4	5
f	3136	2188	1519	1072	781
a_{40}	0	0	0	-8.2×10^{-4}	0
a_{41}	2.3×10^{-2}	0	8.6×10^{-3}	0	1.0×10^{-2}
a_{42}	0	-5.2×10^{-2}	0	0	0
a_{43}	5.0×10^{-3}	0	-7.5×10^{-2}	0	9.8×10^{-2}
a_{44}	0	0	0	5.75×10^{-2}	0
a_{45}	2.7×10^{-3}	0	4.2×10^{-3}	0	-5.5×10^{-2}
a_{46}	0	7.1×10^{-3}	0	0	0
a_{47}	1.6×10^{-3}	0	2.6×10^{-3}	0	2.0×10^{-3}
b_{48}	0	0	0	-6.9×10^{-3}	0
b_{40}	0	0	0	0	0
b_{41}	-3.8×10^{-1}	0	-1.9×10^{-4}	0	3.3×10^{-3}
b_{42}	0	-3.7×10^{-1}	0	0	0
b_{43}	1.7×10^{-2}	0	-3.0×10^{-1}	0	1.1×10^{-3}
b_{44}	0	0	0	2.4×10^{-1}	0
b_{45}	8.3×10^{-3}	0	2.3×10^{-3}	0	-2.0×10^{-1}
b_{46}	0	1.0×10^{-3}	0	0	0
b_{47}	5.4×10^{-3}	0	1.8×10^{-3}	0	-1.9×10^{-3}
b_{48}	0	0	0	-9.5×10^{-3}	0
c_{40}	0	0	0	-2.9×10^{-3}	0
c_{41}	1.0	0	1.1×10^{-2}	0	5.6×10^{-3}
c_{42}	0	1.0	0	0	0
c_{43}	-5.5×10^{-2}	0	1.0	0	-2.6×10^{-2}
c_{44}	0	0	0	1.0	0
c_{45}	-4.3×10^{-2}	0	-6.4×10^{-2}	0	1.0
c_{46}	0	-5.4×10^{-2}	0	0	0
c_{47}	-3.8×10^{-2}	0	-1.1×10^{-3}	0	1.7×10^{-2}
c_{48}	0	0	0	-6.6×10^{-2}	0

Table 4. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Antisymmetric)

m=2 n=	1	2	3	4	5
f(cps)	1829	546	243	187	238
c_{20}	0	0	0	0	0
c_{21}	0	1.0	-1.92×10^{-4}	0	-2.36×10^{-3}
c_{22}	0	1.0	0	0	0
c_{23}	-4.19×10^{-2}	0	1.0	0	-4.53×10^{-2}
c_{24}	0	0	0	1.0	0
c_{25}	-3.71×10^{-2}	0	-1.75×10^{-3}	0	1.0
c_{26}	0	-1.30×10^{-2}	0	0	0
c_{27}	-3.37×10^{-2}	0	-3.23×10^{-3}	0	5.42×10^{-2}
c_{28}	0	0	0	-3.09×10^{-2}	0
c_{29}	-3.07×10^{-2}	0	-4.44×10^{-3}	0	5.93×10^{-2}
$c_2(10)$	0	-9.08×10^{-3}	0	0	0
$c_2(11)$	-2.81×10^{-2}	0	-6.72×10^{-3}	0	1.30×10^{-2}
$c_2(12)$	0	0	0	4.72×10^{-2}	0
$c_2(13)$	-2.58×10^{-2}	0	-1.97×10^{-2}	0	7.77×10^{-3}
$c_2(14)$	0	-1.01×10^{-2}	0	0	0
$c_2(15)$	-2.40×10^{-2}	0	1.16×10^{-2}	0	4.97×10^{-3}
$c_2(16)$	0	0	0	8.50×10^{-3}	0
$c_2(17)$	-2.28×10^{-2}	0	3.62×10^{-3}	0	3.35×10^{-3}
$c_2(18)$	0	-8.91×10^{-2}	0	0	0
$c_2(19)$	-2.25×10^{-2}	0	1.87×10^{-3}	0	2.34×10^{-3}
$c_2(20)$	0	0	0	3.69×10^{-3}	0
$c_2(21)$	-2.39×10^{-2}	0	1.14×10^{-3}	0	1.70×10^{-3}
$c_2(22)$	0	5.13×10^{-3}	0	0	0
$c_2(23)$	-3.07×10^{-2}	0	7.60×10^{-4}	0	1.26×10^{-3}
$c_2(24)$	0	0	0	2.01×10^{-3}	0
$c_2(25)$	-1.29×10^{-1}	0	4.82×10^{-5}	0	9.64×10^{-4}
$c_2(26)$	0	1.78×10^{-3}	0	0	0

Table 4. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Antisymmetric) (Continued)

m=2 n	6	7	8	9	10
f	189	187	204	238	288
	0	0	0	0	0
c_{21}	0	-1.38×10^{-3}	0	-7.32×10^{-4}	0
c_{22}	4.68×10^{-4}	0	0	0	1.12×10^{-3}
c_{23}	0	-2.41×10^{-2}	0	-1.26×10^{-2}	0
c_{24}	0	0	9.33×10^{-3}	0	0
c_{25}	0	-1.30×10^{-1}	0	-1.35×10^{-1}	0
c_{26}	1.0	0	0	0	-4.76×10^{-2}
c_{27}	0	1.0	0	-7.55×10^{-2}	0
c_{28}	0	0	1.0	0	0
c_{29}	0	-8.9×10^{-3}	0	1.0	0
$c_2(10)$	1.90×10^{-3}	0	0	0	1.0
$c_2(11)$	0	2.79×10^{-3}	0	4.64×10^{-2}	0
$c_2(12)$	0	0	1.30×10^{-2}	0	0
$c_2(13)$	0	2.77×10^{-3}	0	1.93×10^{-2}	0
$c_2(14)$	4.64×10^{-3}	0	0	0	2.12×10^{-2}
$c_2(15)$	0	2.07×10^{-3}	0	1.06×10^{-2}	0
$c_2(16)$	0	0	4.53×10^{-3}	0	0
$c_2(17)$	0	1.50×10^{-3}	0	6.50×10^{-3}	0
$c_2(18)$	2.00×10^{-3}	0	0	0	8.15×10^{-3}
$c_2(19)$	0	1.10×10^{-3}	0	4.30×10^{-3}	0
$c_2(20)$	0	0	2.21×10^{-3}	0	0
$c_2(21)$	0	8.16×10^{-4}	0	2.30×10^{-3}	0
$c_2(22)$	1.07×10^{-3}	0	0	0	4.22×10^{-3}
$c_2(23)$	0	6.20×10^{-4}	0	2.17×10^{-3}	0
$c_2(24)$	0	0	1.25×10^{-3}	0	0
$c_2(25)$	0	4.8×10^{-4}	0	1.62×10^{-3}	0
$c_2(26)$	6.39×10^{-4}	0	0	0	2.50×10^{-4}

Table 5. Theoretical and Experimental Radial Frequencies for Eight Equally-Spaced Internal Stringers on a Simply Supported Cylinder

m=1	f in cps				m=2	f in cps			
n	a	b	c	d	n	a	b	c	d
1	737	737	*	*	1	1918	1918	*	*
2	313	313	*	*	2	970	970	*	*
3	160	160	*	*	3	556	556	*	*
4	109	99	104	97	4	359	331	358	*
5	91	92	92	86	5	250	250	253	238
6	105	105	108	103	6	203	203	200	192
7	133	133	140	133	7	191	200	188	182
8	163	178	181	166	8	214	203	208	198
9	216	216	229	217	9	234	234	242	233
10	260	260	282	259	10	276	276	291	281
11	315	315	340	325	11	323	323	346	333
12	368	385	405	*	12	390	391	404	*

m=3					m=4				
1	2587	2587	*	*	1	3136	3136	*	*
2	1622	1622	*	*	2	2188	2168	*	*
3	1047	1047	*	*	3	1519	1519	*	*
4	724	668	*	*	4	1072	1138	*	*
5	504	504	505	*	5	781	781	*	*
6	392	392	380	367	6	639	639	*	*
7	337	337	309	311	7	488	488	495	495
8	304	269	282	*	8	425	522	415	*
9	288	288	297	297	9	398	398	380	398
10	317	317	322	321	10	402	401	387	415
11	356	356	369	370	11	412	412	415	435
12	407	403	422	425	12	493	472	458	*

Key

a = Symmetric modes, stiffened (Donnell's theory).
b = Antisymmetric modes, stiffened (Donnell's theory).
c = Unstiffened, experimental results (Schnell and Heinrichsbauer).
d = Stiffened experimental results (Schnell and Heinrichsbauer).

Type of Stringer: (Hat-Shaped).

Table 6. Frequencies of Vibration for Radial Modes (Four Stringers)

m / n	f in cps 1	2	3	4
1	745	1829	2648	3136
2	317	984	1637	2188
3	157	546	1039	1519
4	105	356	721	1072
5	91	143	491	781
6	106	204	396	639
7	137	187	313	488
8	173	207	302	424
9	222	238	288	384
10	265	281	321	402
11	330	337	362	412
12	386	395	418	493
13	460	463	476	510
14	529	531	544	596
15	610	616	623	639
16	691	698	691	724
17	785	790	792	808
18	878	873	885	898
19	977	984	981	995
20	1085	1093	1096	1201
21	1194	1202	1209	1209
22	1361	1330	1324	1330
23	1434	1443	1449	1447
24	1571	1581	1587	1590
25	1699	1709	1715	1717

Table 7. Frequencies of Vibration for Radial Modes (Eight Stringers)

m / n	f in cps 1	2	3	4
1	737	1918	2587	3136
2	313	970	1622	2188
3	160	556	1047	1519
4	109	359	724	1072
5	92	250	504	782
6	105	203	392	639
7	132	191	337	488
8	163	214	304	425
9	215	234	288	398
10	260	276	317	402
11	315	323	356	412
12	368	390	407	493
13	449	451	466	499
14	512	513	526	530
15	570	576	588	612
16	653	659	674	702
17	774	770	777	788
18	837	833	845	856
19	907	912	905	928
20	1040	1044	1055	1068
21	1173	1176	1185	1189
22	1247	1252	1252	1262
23	1336	1336	1342	1351
24	1507	1511	1517	1529
25	1680	1682	1689	1696

Table 8. Frequencies of Vibration for Longitudinal Modes
(Four Stringers)

m / n	f in cps 1	2	3	4	m / n	f in cps 1	2	3	4
1	3275	4562	5579	6637	14	21438	21628	21848	22541
2	5014	6016	7152	8265	15	23124	23300	23498	23928
3	5943	6979	8764	9680	16	25289	25645	26106	26840
4	7328	7946	8878	11075	17	26226	26375	26502	26885
5	9294	9844	10605	11557	18	27761	27916	28051	28413
6	9441	10198	11436	13160	19	28442	28758	29150	29865
7	11957	12313	12808	13508	20	31041	31165	31233	31517
8	12857	13486	14455	15547	21	32581	32715	32793	33103
9	13872	14232	14731	15846	22	33457	33715	33990	34571
10	16675	16923	17243	17769	23	35875	35978	35993	36235
11	17178	17593	18188	19063	24	36438	36691	36944	37518
12	18423	18739	19182	19946	25	37308	37430	37468	37753
13	20546	20966	21555	22228					

Table 9. Frequencies of Vibration for Circumferential Modes
(Four Stringers)

m / n	f in cps 1	2	3	4	m / n	f in cps 1	2	3	4
1	46993	47070	46991	47175	14	66825	66919	66797	67072
2	49920	50110	50200	50629	15	68960	69097	69020	69338
3	50492	50564	50461	50630	16	70272	70322	70096	70214
4	51900	51975	51869	52043	17	71736	71785	71550	71663
5	52612	52790	52851	53261	18	74695	74824	74906	75002
6	55398	55464	55328	55480	19	77184	77308	77372	77459
7	56821	56890	56707	56914	20	82961	83073	83092	83151
8	58170	58326	58327	58685	21	85439	85551	85591	85620
9	60322	60380	60212	60350	22	91232	91335	91335	91337
10	60767	60924	60913	61272	23	93724	93826	93822	93821
11	61741	61807	61645	61800	24	99504	99597	99567	99527
12	65272	65327	65129	65256	25	102041	102136	102084	102062
13	66418	66521	66379	66559					

Table 10. Comparison of Flügge's Theory versus Donnell's Theory for Four Equally Spaced Stringers on Simply Supported Cylinder

m=1		f in cps			m=2		f in cps		
n	a	b	c	d	n	a	b	c	d
1	756.2	745	*	*	1	1945	1829	*	*
2	332	317	*	*	2	997	984	*	*
3	164.3	157	170	156	3	602	546	*	*
4	105.1	105	104	100	4	356	356	358	336
5	92.5	91.5	92	89	5	254	243	253	229
6	108.2	106.1	108	104	6	202	204	200	194
7	140	137	140	137	7	187	187	188	187
8	173	173	181	174	8	211	207	208	207
9	228	222	229	224	9	236	238	242	238
10	279	265	282	265	10	295	281	291	285
11	335	330	340	336	11	342	337	346	336
12	410	386	405	396	12	421	395	404	401

m=3					m=4				
1	2650	2648	*	*	1	3341	3136	*	*
2	1691	1637	*	*	2	2316	2188	*	*
3	1137	1039	*	*	3	1603	1519	*	*
4	688	721	*	*	4	1089	1072	*	*
5	491	491	460	*	5	793	781	*	*
6	371	396	380	372	6	630	639	*	*
7	313	313	309	305	7	482	488	495	482
8	297	302	282	*	8	441	424	415	420
9	302	288	297	296	9	385	384	380	385
10	344	321	322	329	10	411	402	387	392
11	386	362	369	372	11	416	412	415	425
12	443	418	422	421	12	491	493	458	509

<u>Key</u>

a = Symmetric modes (Flügge's theory).
b = Symmetric modes (Donnell's theory).
c = Unstiffened experimental.
d = Experimental results for stiffened shell [4].

Type of Stringer: (Hat-shaped).

* = No experimental results for this case.

Table 11. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

m=1 / n	a	b	c	d	m=1 / n	a	b	c	d
	f in cps					f in cps			
1	740	737	740	737	14	488	512	488	512
2	323	313	323	313	15	525	570	525	570
3	162	160	162	160	16	691	653	699	657
4	106	109	102	99	17	780	774	780	774
5	90.6	92	90.6	92	18	821	837	821	837
6	105	105	105	105	19	822	907	822	907
7	132	132	132	132	20	1108	1040	1163	1181
8	170	163	182	178	21	1213	1173	1213	1173
9	217	215	217	216	22	1230	1247	1230	1247
10	256	260	256	260	23	1283	1336	1283	1336
11	307	315	307	315	24	1605	1507	1536	1403
12	390	368	404	385	25	1762	1680	1762	1680
13	452	449	452	449					

Table 12. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

m=2 / n	a	b	c	d	m=2 / n	a	b	c	d
	f in cps					f in cps			
1	1921	1918	1921	1918	14	491	513	491	513
2	1005	970	1005	970	15	579	577	579	577
3	572	556	572	556	16	701	659	703	661
4	347	359	335	331	17	788	770	788	770
5	241	250	241	250	18	804	833	804	833
6	195	203	195	203	19	825	912	825	912
7	181	191	187	200	20	1114	1044	1079	1003
8	210	214	206	203	21	1218	1176	1218	1176
9	236	234	236	234	22	1234	1252	1242	1257
10	273	276	273	276	23	1284	1336	1284	1336
11	313	323	313	323	24	1611	1511	1526	1406
12	415	390	415	391	25	1769	1682	1769	1682
13	455	451	455	451					

a = Symmetric modes (Flügge's theory).
b = Symmetric modes (Donnell's theory).
c = Antisymmetric modes (Flügge's theory).
d = Antisymmetric modes (Donnell's theory).

Table 13. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

m=3	f in cps				m=3	f in cps			
n	a	b	c	d	n	a	b	c	d
1	2734	2587	2734	2587	14	510	526	513	528
2	1701	1622	1701	1622	15	542	588	542	588
3	1089	1047	1089	1047	16	706	674	696	668
4	723	724	668	668	17	786	777	786	777
5	515	504	515	504	18	815	845	823	853
6	379	392	379	392	19	849	905	849	905
7	315	337	315	337	20	1131	1055	1104	1010
8	309	304	275	269	21	1225	1185	1225	1185
9	282	288	282	288	22	1240	1252	1266	1278
10	300	317	300	317	23	1292	1342	1292	1342
11	324	356	324	356	24	1613	1517	1643	1626
12	422	407	406	403	25	1764	1689	1764	1689
13	462	466	462	466					

a = Symmetric modes (Flügge's theory).
b = Symmetric modes (Donnell's theory).
c = Antisymmetric modes (Flügge's theory).
d = Antisymmetric modes (Donnell's theory).

Table 14. Comparison of Donnell's Theory versus "Orthotropic Theory" for Four Equally-Spaced Stringers on Simply-Supported Cylinder

m=1	f in cps			m=2	f in cps		
n	a	b	c	n	a	b	c
1	745	745	*	1	1829	1829	*
2	317	302	304	2	984	959	952
3	157	157	154	3	546	546	537
4	105	99	100	4	356	357	338
5	91	91	91	5	243	243	238
6	106	109	107	6	204	189	194
7	137	137	138	7	187	187	187
8	173	181	177	8	207	204	206
9	222	222	223	9	238	238	241
10	265	279	275	10	281	286	288
11	330	330	332	11	337	337	342
12	386	386	395	12	395	401	403
m=3				m=3			
1	2648	2648	*	1	3136	3136	*
2	1637	1647	1618	2	2188	2168	2717
3	1039	1039	1018	3	1519	1519	1494
4	721	673	677	4	1072	816	1055
5	491	491	482	5	781	781	777
6	396	355	370	6	639	592	602
7	313	313	312	7	488	488	495
8	302	273	292	8	424	391	436
9	288	288	296	9	384	384	412
10	321	315	330	10	402	371	416
11	362	362	374	11	412	412	441
12	418	418	428	12	493	455	482

<u>Key</u>

a = Symmetric modes, stiffened.
b = Antisymmetric modes, stiffened.
c = Orthotropic results of Schnell and Heinrichsbauer

Type of Stringer: _⊓_ (Hat-Shaped).

* = No experimental results for this case.

Table 15. Comparison of Theoretical Radial Frequencies for Four Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

f in cps

	m=1		m=2		m=3		m=4	
n	a	b	a	b	a	b	a	b
1	745	745	1829	1831	2648	2649	3136	3138
2	317	318	984	985	1637	1642	2188	2189
3	157	161	546	549	1039	1046	1519	1537
4	105	108	356	358	721	728	1072	1089
5	91	91	243	245	491	523	781	787
6	106	107	204	213	396	400	639	643
7	137	137	187	189	313	332	488	492
8	173	173	207	216	302	307	424	440
9	222	222	238	241	288	291	384	391
10	265	266	281	286	321	343	402	413
11	330	331	337	337	362	374	412	418
12	386	387	395	400	418	432	493	501
13	460	461	449	461	476	488	510	522
14	529	531	512	529	544	549	596	602
15	610	615	570	619	623	627	639	648

a = Symmetric modes (Donnell's theory).

b = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).

Table 16. Comparison of Theoretical Longitudinal Frequencies for Four Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

m=2 n	f in cps a	b	m=3 n	f in cps a	b
1	4562	4569	1	5579	5583
2	6016	6091	2	7152	7165
3	6979	6994	3	8764	8799
4	7946	7963	4	8878	8892
5	9844	9881	5	10605	10614
6	10198	1133	6	11436	11486
7	12313	12360	7	12808	12841
8	13486	13498	8	14455	14472
9	14232	14275	9	14731	14756
10	16923	16982	10	17243	17268
11	17593	17654	11	18188	18209
12	18739	18775	12	19182	19197
13	20966	21036	13	21555	21578
14	21628	21680	14	21848	21894
15	23300	23336	15	23498	23533
16	25645	25689	16	26106	26135
17	26375	26403	17	26502	26514
18	27916	28038	18	28051	28082
19	28758	28841	19	29150	29158
20	31165	31198	20	31233	31252

a = Symmetric modes (Donnell's theory).

b = Symmetric modes (Donnell's theory but neglecting rotatory inertia).

Table 17. Comparison of Theoretical Circumferential Frequencies for Four Equally-Spaced Stringers Neglecting Rotatory Inertia of Stringers

m=2	f in cps		m=3	f in cps	
n	a	b	n	a	b
1	47070	48881	1	46991	47116
2	50110	51017	2	50200	50203
3	50564	51307	3	50461	50535
4	51975	52608	4	51869	52213
5	52790	53751	5	52851	52887
6	55464	56183	6	55328	55432
7	56890	58084	7	56707	56994
8	58326	58622	8	58327	58387
9	60380	60702	9	60212	60380
10	60924	61069	10	60913	60964
11	61807	62121	11	61645	61679
12	65327	66286	12	65129	65231
13	66521	66813	13	66379	66505
14	66919	68870	14	66797	66838
15	69097	71405	15	69020	69091
16	70322	71270	16	70096	70115
17	71785	74496	17	71550	74688
18	74824	77068	18	74906	77267
19	77308	82715	19	77372	82888
20	83073	85306	20	83092	85489

Table 18. Comparison of Theoretical Radial Frequencies of the Present Analysis versus the Analysis of Reference [6] for 60 Stringers

m=1 n	f in cps a	b	m=2 n	f in cps a	b
1	1128	1398	1	1845	1950
2	668	737	2	1342	1428
3	424	445	3	1001	1059
4	293	304	4	840	824
5	220	232	5	628	675
6	181	198	6	530	580
7	165	188	7	456	520
8	167	196	8	410	486
9	182	216	9	383	471
10	207	245	10	371	471
11	239	281	11	373	483
12	278	323	12	387	504
13	321	369	13	411	535
14	369	420	14	443	572
15	421	475	15	483	616

a = Results of present analysis (Donnell's theory).

b = Results of Reference [6].

Type of Stringer: .

Table 19. Comparison of Theoretical Radial Frequencies for Eight Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

m=1	f in cps		m=2	f in cps	
n	a	b	n	a	b
1	737	740	1	1918	1952
2	313	316	2	970	981
3	160	162	3	556	563
4	109	113	4	359	364
5	92	93	5	250	252
6	105	107	6	203	207
7	132	133	7	191	199
8	163	166	8	214	223
9	215	215	9	234	238
10	260	263	10	276	281
11	315	316	11	323	325
12	368	368	12	396	397
13	449	449	13	451	454
14	512	514	14	513	514
15	570	572	15	577	579
16	653	653	16	659	659
17	774	775	17	770	770
18	837	845	18	833	835
19	907	920	19	912	927
20	1040	1044	20	1044	1050
21	1173	1181	21	1176	1184
22	1247	1272	22	1252	1276
23	1336	1337	23	1336	1339
24	1507	1507	24	1526	1542
25	1680	1680	25	1769	1788

a = Symmetric modes (Donnell's theory).

b = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).

Table 20. Comparison of Theoretical Radial Frequencies for Eight Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

m=3 n	f in cps a	b	m=4 n	f in cps a	b
1	2587	2582	1	3136	3171
2	1622	1686	2	2188	2204
3	1047	1071	3	1519	1532
4	724	771	4	1072	1142
5	504	517	5	782	792
6	392	420	6	639	655
7	337	351	7	488	492
8	304	306	8	425	429
9	288	291	9	398	403
10	317	342	10	402	416
11	356	368	11	412	424
12	407	421	12	493	507
13	466	489	13	499	518
14	526	589	14	530	554
15	588	625	15	612	634
16	674	672	16	702	711
17	777	832	17	788	803
18	845	866	18	856	881
19	905	1022	19	928	940
20	1055	1185	20	1068	1189
21	1185	1240	21	1189	1265
22	1252	1344	22	1262	1355
23	1342	1468	23	1351	1468
24	1517	1580	24	1529	1697
25	1689	1767	25	1696	1767

a = Symmetric modes (Donnell's theory).

b = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).

Table 21. Comparison of Present Theory versus Experimental Results of Hoppman [1] for 16 Equally-Spaced Stringers

m=1			f in cps			m=2			f in cps		
n	a	b	c	d	e	n	a	b	c	d	e
1	1493	*	*	*	1493	1	3868	*	*	*	3868
2	819	700	750	742	819	2	2202	*	2300	1880	2202
3	1261	1270	1150	1330	1261	3	1788	1830	1700	1740	1789
4	2153	2200	2100	2480	2153	4	2406	2600	2350	2680	2406
5	3340	3460	3340	4060	3341	5	3519	4080	3510	4120	3514
m=3						m=4					
1	6100	*	*	*	6101	1	7854	*	*	*	7854
2	3654	*	4200	*	3654	2	5164	*	6100	*	5164
3	2741	2640	2870	2470	2742	3	3880	5490	4360	*	3880
4	2945	3360	2970	3040	2995	4	3726	4100	3960	3710	3726
5	3868	4120	3900	4340	3868	5	4417	5130	4620	4780	4417

* = No experimental data given.

a = Donnell's theory (stiffened shell).

b = Experimental results (stiffened shell).

c = Orthotropic results (stiffened shell).

d = Experimental results (unstiffened shell).

e = Donnell's theory but neglecting rotatory inertia of the stringers.

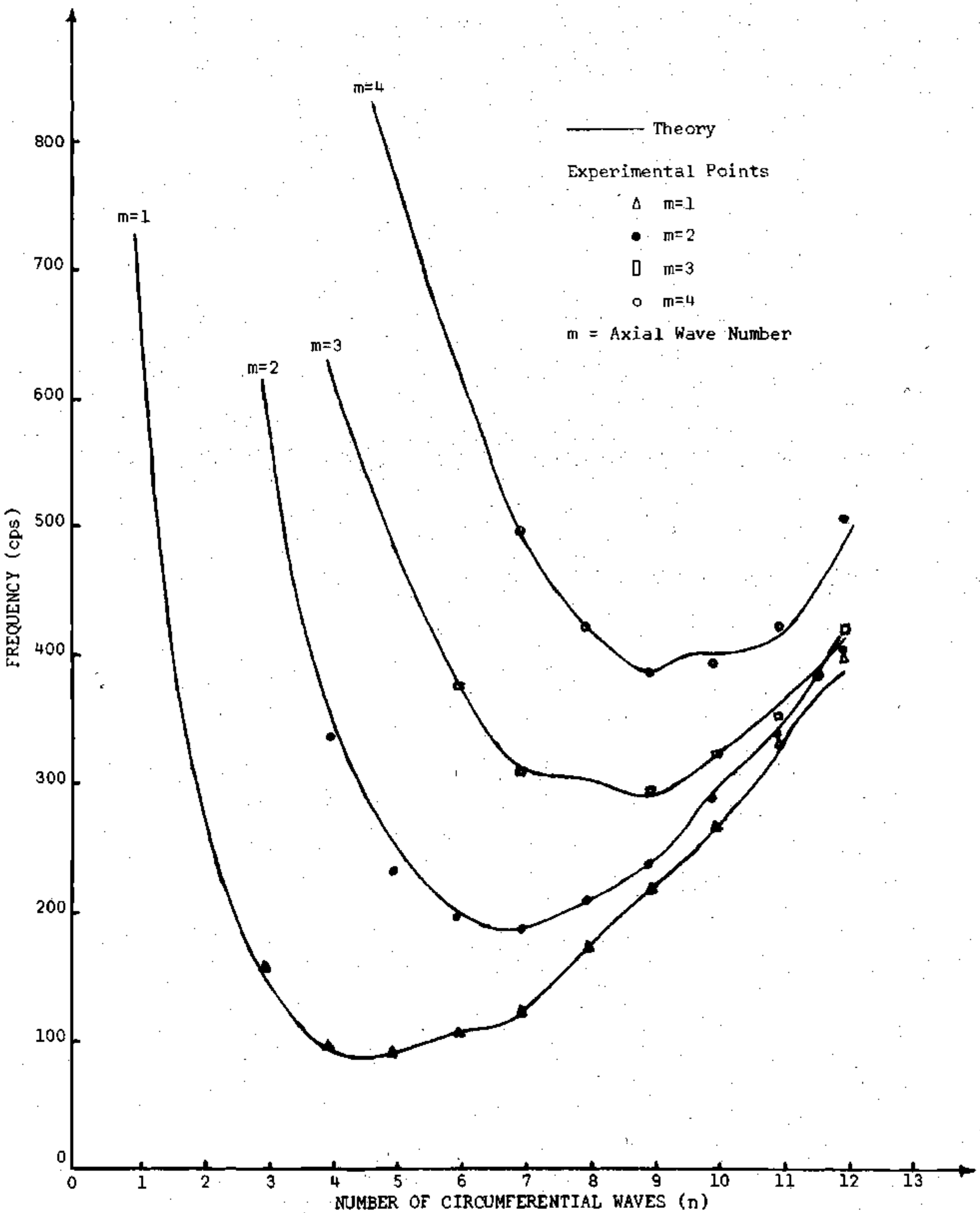


Figure 3. Radial Frequencies of Stiffened Cylinder with Four Stringers

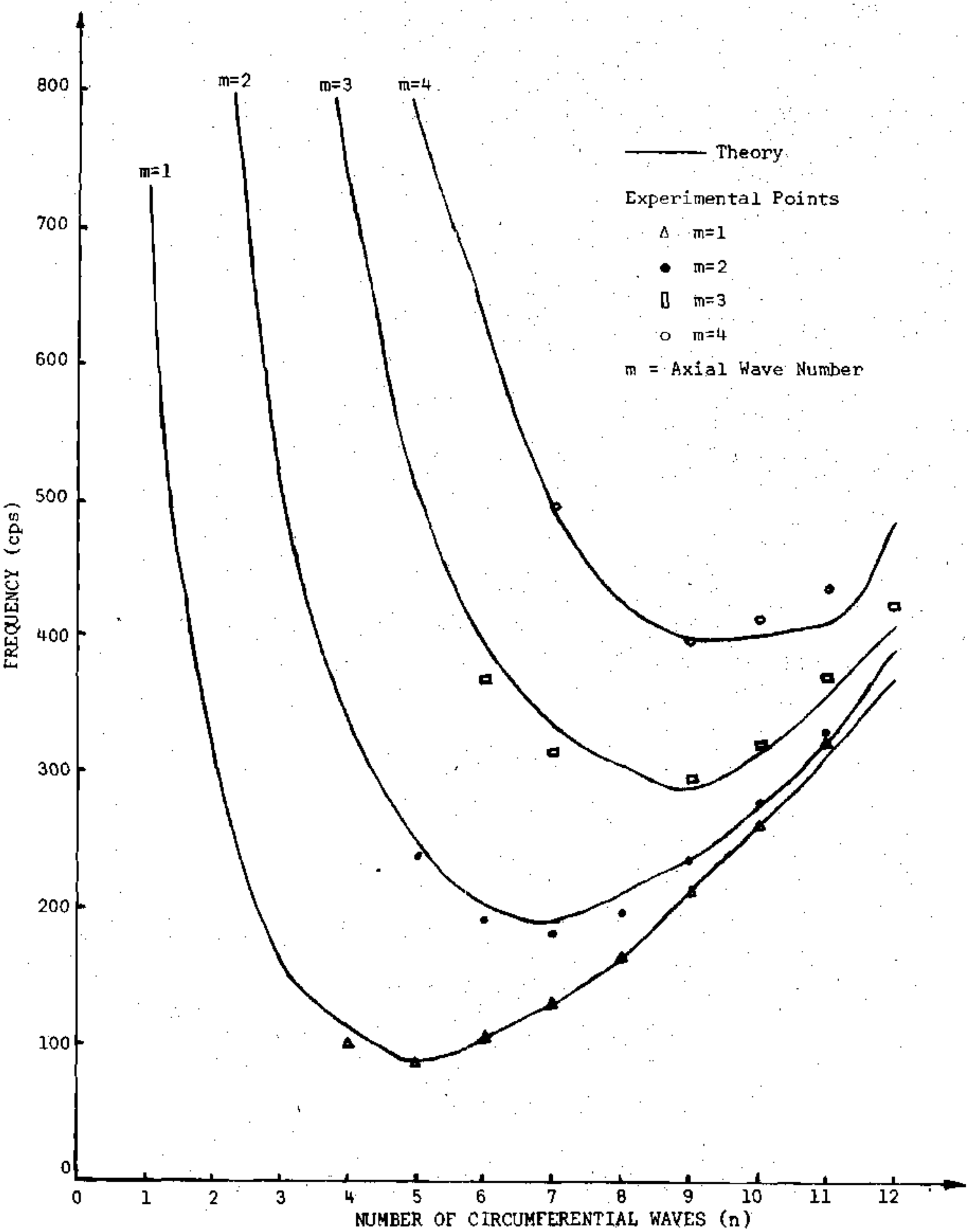


Figure 4. Radial Frequencies of Stiffened Cylinder with Eight Stringers

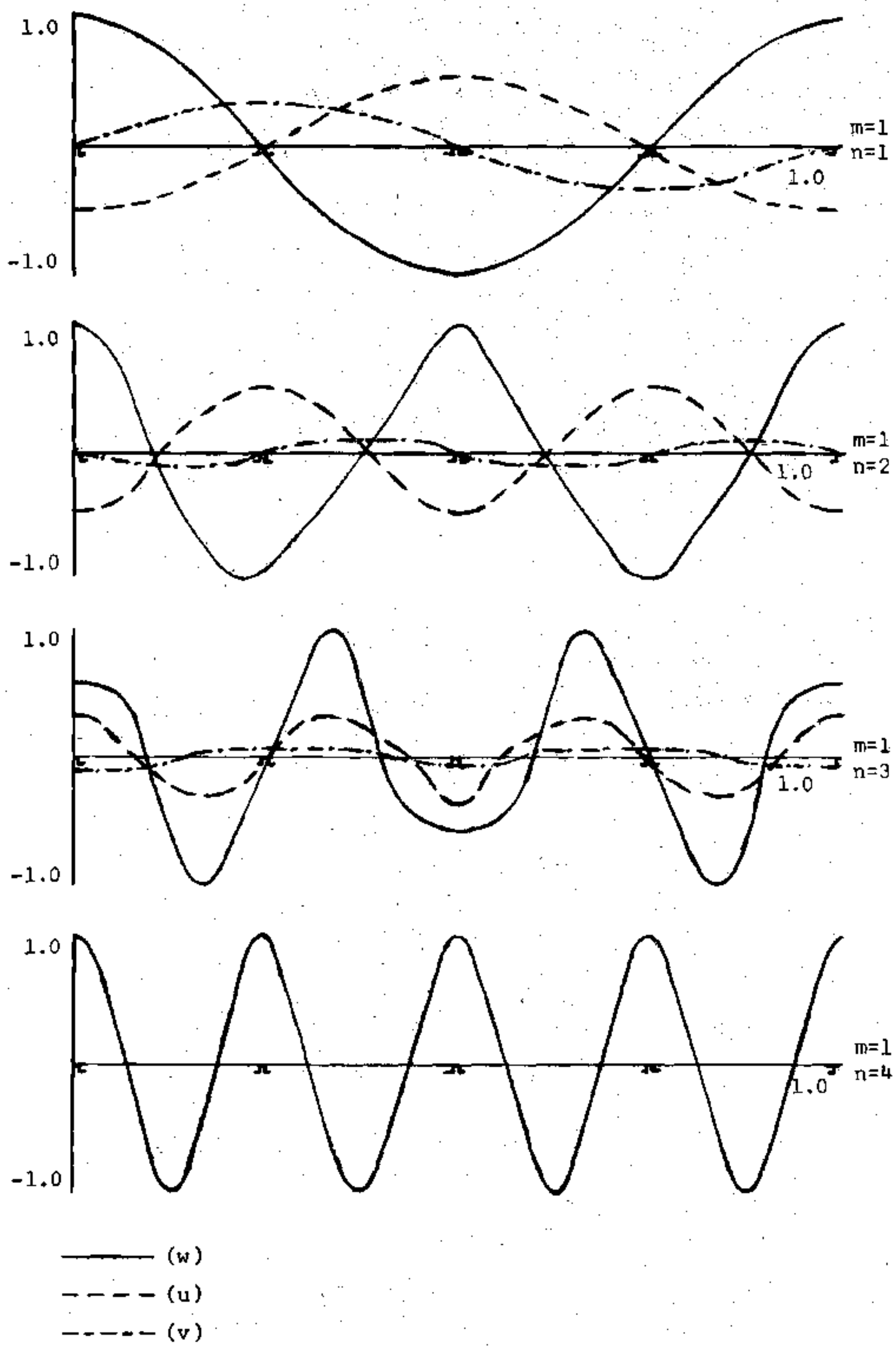


Figure 5(a). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric)

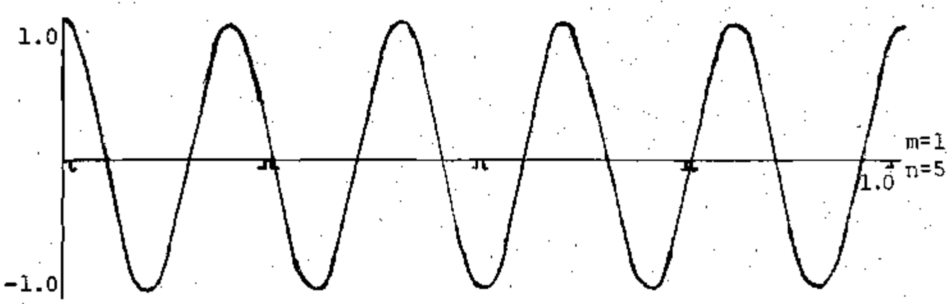


Figure 5(a). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric) (Continued)

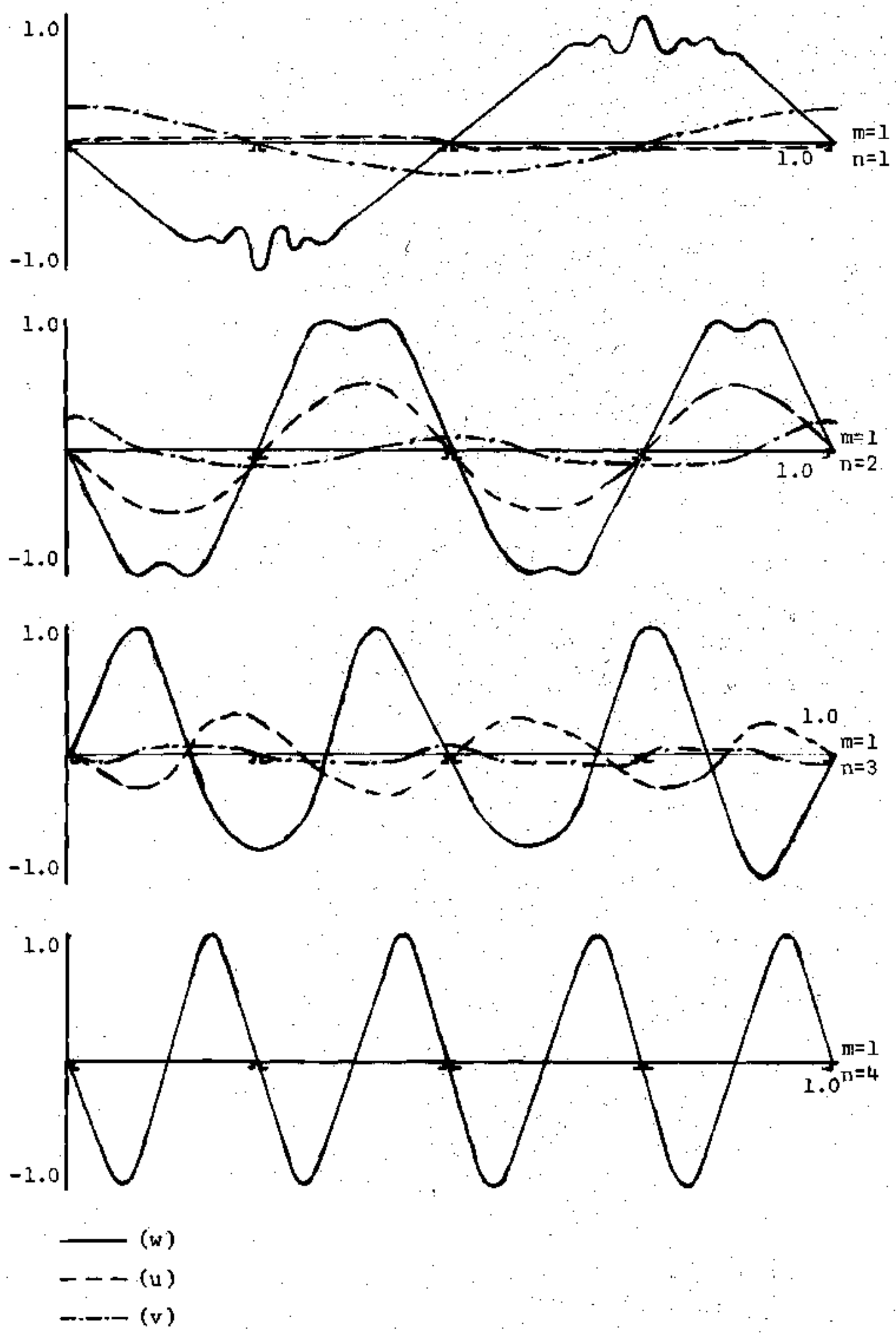


Figure 5(b). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Antisymmetric)

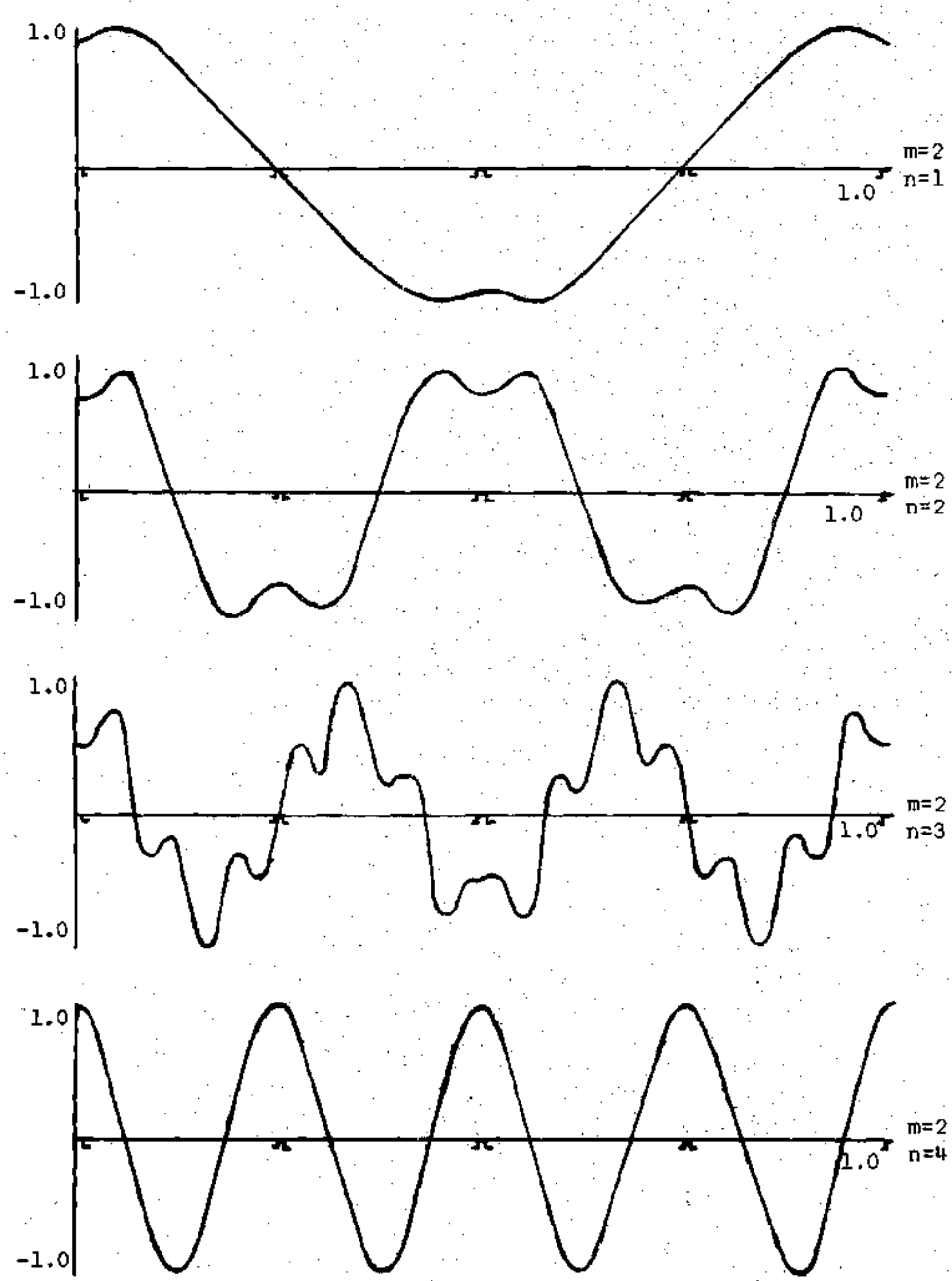


Figure 6. Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric)

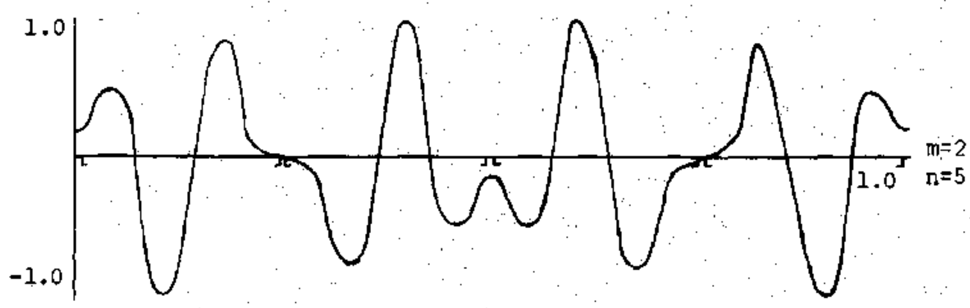


Figure 6. Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric) (Continued)

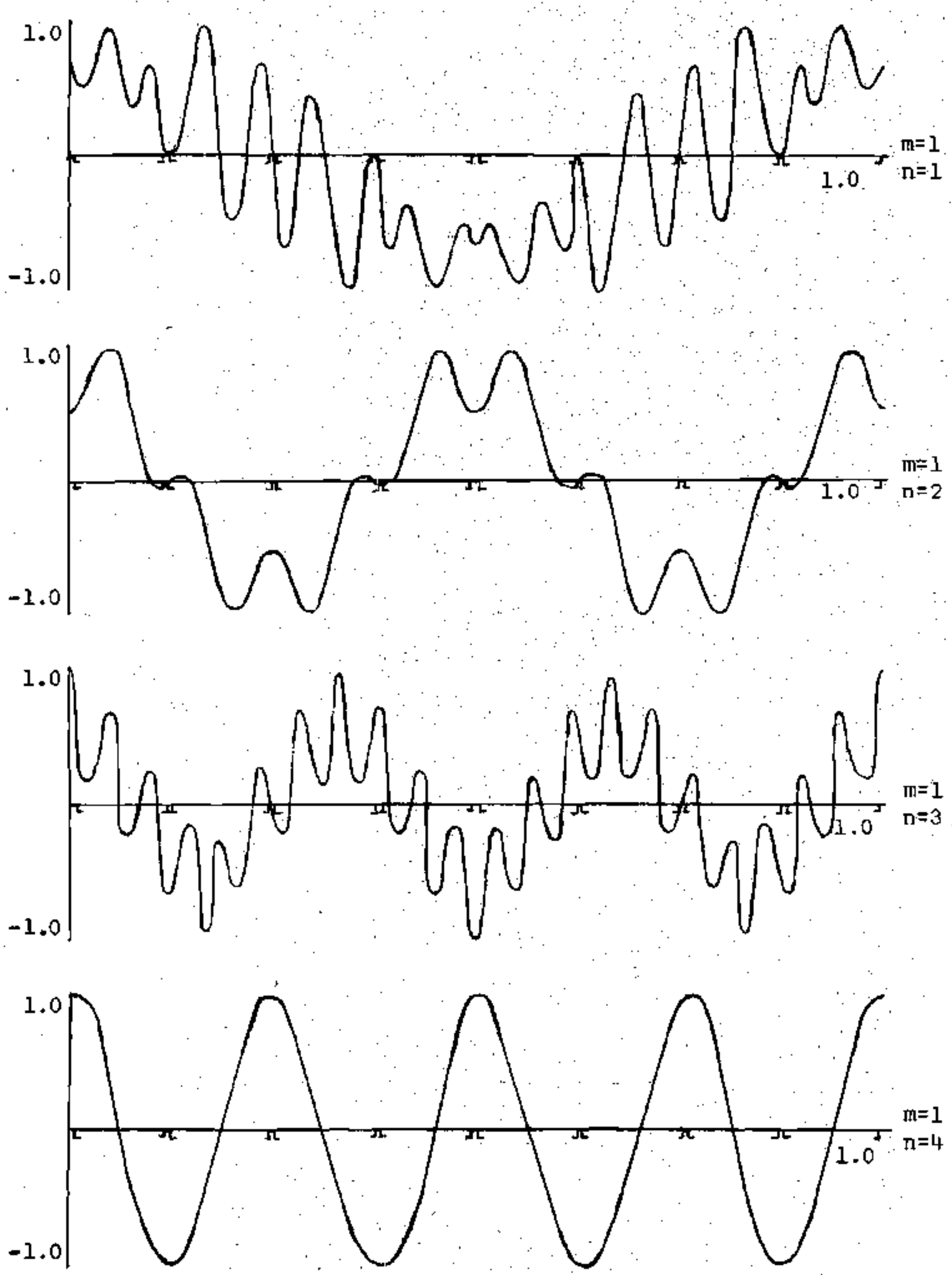


Figure 7. Theoretical Mode Shapes for Stiffened Shell with Eight Stringers (Symmetric)

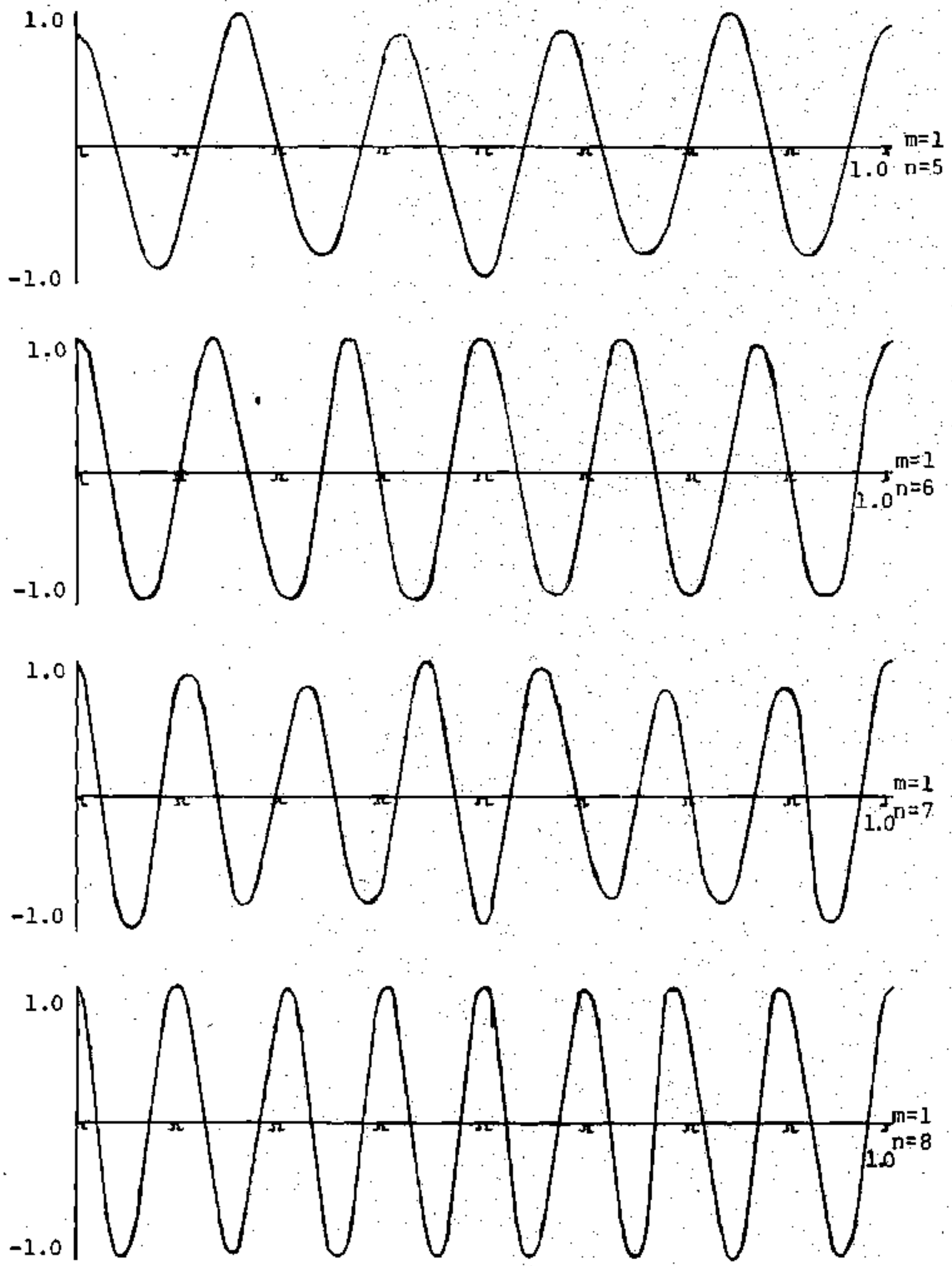


Figure 7. Theoretical Mode Shapes for Stiffened Shell with Eight Stringers (Symmetric) (Continued)

CHAPTER IV

GOVERNING DIFFERENTIAL EQUATIONS OF MOTION AND ANALYSIS OF STRINGER STIFFENED CYLINDRICAL SHELLS

In Chapter III the free vibration of a stringer stiffened cylindrical shell with simple supports was analyzed by using the Ritz method. The results of that analysis showed that the use of Donnell's shell theory led to results in good agreement with the more general Flügge's shell theory and with the experimental results cited in the literature but that neglecting the in-plane inertias in calculating the radial frequencies is not a completely justifiable assumption. Furthermore, neglecting rotatory inertia of the shell had no effect on the frequencies and mode shapes of the shell for the axial ($m \leq 4$) and circumferential ($n \leq 30$) modes investigated. Actually, this is to be expected since for an unstiffened thin cylindrical shell when one neglects shear distortion and rotatory inertia of the shell wall the results apply when $\left(\frac{\ell}{m} > 10h, \frac{\pi}{n} > \frac{10h}{a}\right)$ where m is the number of axial half-waves and n is the number of full circumferential waves.

Therefore, Donnell's equations including in-plane inertias but neglecting rotatory inertia and shear distortion of the shell wall are now employed in order to investigate the free vibration of stringer stiffened, cylindrical shells with arbitrary admissible boundary conditions. The present analysis considers the thin cylindrical shell and stringers as discrete structural components.

The stringer-shell interaction loads are represented by double Fourier series with unknown coefficients. The nonhomogenous equations of motion of the cylindrical shell are used to determine the shell solutions for arbitrary admissible boundary conditions. One then obtains the displacements of the shell as functions of the unknown loading coefficients. Similarly, V. Z. Vlasov's equations of motion for the stringers are solved for the stringer displacements in terms of the Fourier loading coefficients for arbitrary admissible boundary conditions.

Once the analysis of the stringer and shell problems have been completed, the kinematical conditions at the line of attachment that require compatible deformations are imposed. This yields an infinite number of equations in the unknown Fourier coefficients for the determination of the eigenfrequencies of the composite structure. Once the frequencies are found, then calculations can be performed for the mode shapes of free vibration.

Equations of Motion for the Shell

The partial differential equations of motion of a thin cylindrical shell including surface forces according to Donnell's approximation for the coordinate system shown in Figure 12, p.106, are:

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2a^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{(1+\nu)}{2a}\frac{\partial^2 v}{\partial x \partial \phi} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{(1-\nu^2)}{Eh}p_x = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2} \qquad (4.1)$$

$$\frac{(1+\nu)}{2a}\frac{\partial^2 u}{\partial x\partial\phi} + \frac{(1-\nu)}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial\phi^2} - \frac{1}{a^2}\frac{\partial w}{\partial\phi} + \frac{(1-\nu^2)}{Eh}p_\phi = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2} \quad (4.2)$$

$$\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial\phi} - \frac{w}{a^2} - \frac{h^2}{12}\nabla^4 w + \frac{(1-\nu^2)}{Eh}p_z = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 w}{\partial t^2} \quad (4.3)$$

where p_x, p_ϕ, and p_z are the applied surface forces. Examination of Equations (4.1), (4.2), and (4.3) shows that each equation is a coupled partial differential equation in the middle surface displacements u, v, and w. In a similar manner as Donnell, these equations can be uncoupled by differentiation and elimination (see Appendix B). The equations that are derived from this uncoupling process are as follows:

$$\nabla^4 u - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1}{a^2}\frac{\partial^3 w}{\partial x\partial\phi^2} = -\frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\left[\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2} + \frac{\nu}{a}\frac{\partial w}{\partial x}\right.$$

$$\left. - \frac{(3-\nu)}{2}\nabla^2_u - \frac{(1-\nu^2)}{Eh}p_x\right] - \frac{(1-\nu^2)}{Eh}\frac{\partial^2 p_x}{\partial x^2} - \frac{2(1+\nu)}{Eha^2}\frac{\partial^2 p_x}{\partial\phi^2} +$$

$$\frac{(1+\nu)^2}{Eha}\frac{\partial^2 p_\phi}{\partial x\partial\phi}, \quad (4.4)$$

$$\nabla^4 v - \frac{(2+\nu)}{a^2}\frac{\partial^3 w}{\partial x^2\partial\phi} - \frac{1}{a^4}\frac{\partial^3 w}{\partial\phi^3} = -\frac{2(1+\nu)}{E}\frac{\partial^2}{\partial t^2}\left[\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2} - \right.$$

$$\frac{(3-\nu)}{2} \quad \nabla v^2 + \frac{1}{a^2}\frac{\partial w}{\partial \phi} - \frac{(1-\nu^2)}{Eh}\, p_\phi \Big] - \frac{(1-\nu^2)}{Eha^2}\frac{\partial^2 p_\phi}{\partial \phi^2} -$$

$$\frac{2(1-\nu^2)}{Eh(1-\nu)}\frac{\partial^2 p_\phi}{\partial x^2} + \frac{(1-\nu^2)(1+\nu)}{Eha(1-\nu)}\frac{\partial^2 p_x}{\partial x \partial \phi}\ , \tag{4.5}$$

$$\frac{h^2}{12}\nabla w^8 + \frac{(1-\nu^2)}{a^2}\frac{\partial^4 w}{\partial x^4} - \frac{(1-\nu^2)}{Eh}\nabla^4 p_z = -\frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\Bigg[\ \{$$

$$\frac{(1-\nu^2)\rho}{E}\frac{\partial^2}{\partial t^2} - \frac{(3-\nu)}{2}\nabla^2\} \quad \{\frac{\rho(1-\nu^2)}{E}\frac{\partial^2 w}{\partial t^2} + \frac{w}{a^2} + \frac{h^2}{12}\nabla^4 w$$

$$-\frac{(1-\nu^2)}{Eh}\, p_z\} + \frac{(1-\nu)}{2}\nabla^4 w + \frac{\nu^2}{a^2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4}\frac{\partial^2 w}{\partial \phi^2}\Bigg] -$$

$$-\frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\left[-\frac{\nu(1-\nu)}{aEh}\frac{\partial p_x}{\partial x} - \frac{(1-\nu^2)}{Eha^2}\frac{\partial p_\phi}{\partial \phi}\right] -$$

$$\frac{\nu(1-\nu^2)}{aEh}\frac{\partial^3 p_x}{\partial x^3} + \quad \frac{(1-\nu^2)}{a^3\ Eh} \quad \frac{\partial^3 p_x}{\partial x \partial \phi^2} -$$

$$\frac{1}{a^4}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial \phi^3} - \frac{(2+\nu)(1-\nu^2)^2}{a^2(1-\nu)Eh}\frac{\partial^3 p_\phi}{\partial x^2 \partial \phi}\ , \tag{4.6}$$

where

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}\right).$$

For the case where $p_x = p_\phi = p_z = 0$, then the above equations reduce to those obtained by Yu [16].

When the stringer is deformed, the interaction loads of the jth stringer may be summed into resultant vertical, longitudinal and tangential forces acting through the shear center and a resultant torque about the shear center. This approach was used by Lin [19] to investigate the random vibration of stringer-stiffened, simply-supported flat plates (i.e., fuselage panels). The resultant forces acting on the jth stringer are replaced by statically equivalent surface loads acting on the shell. The surface loads are then represented by double Fourier series in the circumferential and longitudinal directions. Mathematically, these loadings are given by:

(i) Normal load $p_{2j}^{(1)}(x,\phi)$ (see Figure 8):

$$p_{zj}^{(1)}(x,\phi) = \begin{cases} -\dfrac{z_j(x)}{2\varepsilon_{ij}}, \; x\varepsilon[0,\ell], \; \phi\varepsilon\left[\phi_j - \dfrac{\varepsilon_{1j}}{a}, \; \phi_j + \dfrac{\varepsilon_{1j}}{a}\right] \\ \\ 0, \quad \text{otherwise} \end{cases} \tag{4.7}$$

(ii) Moment $p_{zj}^{(2)}(x,\phi)$ (see Figure 9):

$$p_{zj}^{(2)}(x,\phi) = \begin{cases} \dfrac{Q_j(x)}{\varepsilon_{2j}}\,, & x\varepsilon[0,\ell],\ \phi\varepsilon\left[\phi_j - \dfrac{\varepsilon_{2j}}{a}\,,\ \phi\right] \\ -\dfrac{Q_j(x)}{\varepsilon_{2j}}\,, & x\varepsilon[0,\ell],\ \phi\varepsilon\left[\phi_j,\ \phi_j + \dfrac{\varepsilon_{2j}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \tag{4.8}$$

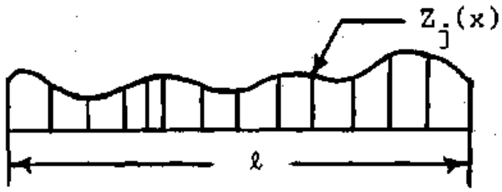
(iii) Tangential load $p_{\phi j}(x,\phi)$ (see Figure 10):

$$p_{\phi j}(x,\phi) = \begin{cases} -\dfrac{Y_j(x)}{2\varepsilon_{3j}}\,, & x\varepsilon[0,\ell],\ \phi\varepsilon\left[\phi_j - \dfrac{\varepsilon_{3j}}{a}\,,\ \phi_j + \dfrac{\varepsilon_{3j}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \tag{4.9}$$

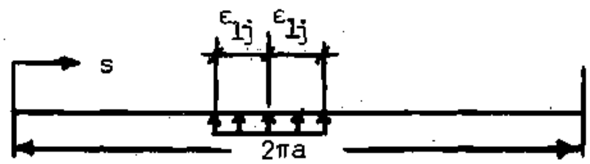
(iv) Longitudinal load $p_{xj}(x,\phi)$ (see Figure 11):

$$p_{xj}(x,\phi) = \begin{cases} -\dfrac{X_j(x)}{2\varepsilon_{aj}}\,, & x\varepsilon[0,\ell],\ \phi\varepsilon\left[\phi_j - \dfrac{\varepsilon_{aj}}{a},\ \phi_j + \dfrac{\varepsilon_{aj}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \tag{4.10}$$

where ϕ_j is the location of the jth stringer.

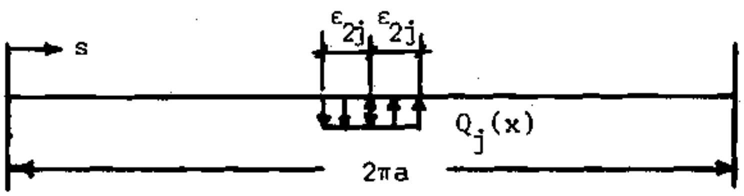


(a) Longitudinal Direction



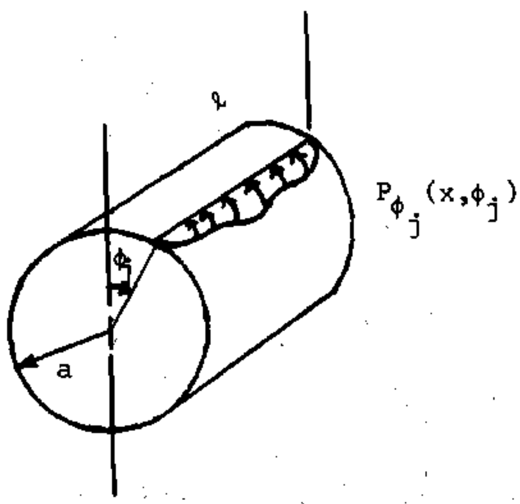
(b) Circumferential Direction

Figure 8. Schematic Sketch of Normal Loading

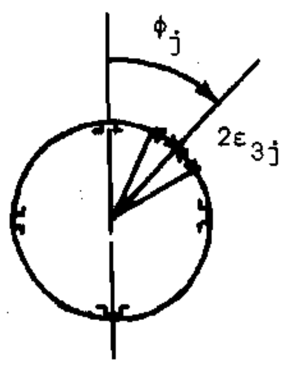


Circumferential Direction

Figure 9. Schematic Sketch of Twisting Moment



(a) Longitudinal Direction



(b) Circumferential Direction

Figure 10. Schematic Sketch of Tangential Shear Loading

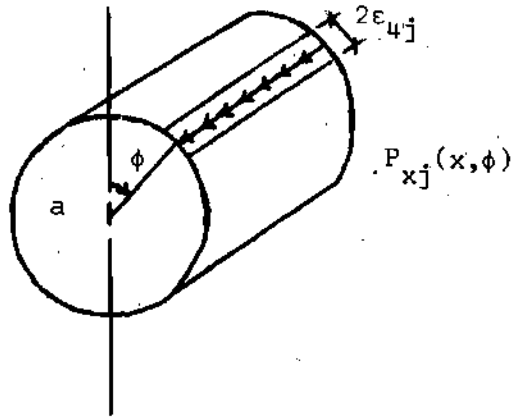


Figure 11. Schematic Sketch of Longitudinal Shear Loading

We assume that the distributed loads which act upon the shell can be expanded into double Fourier series as follows:

$$p_{zj}^{(1)}(x,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(Z_{jmn}\cos n\phi + Z'_{jmn}\sin n\phi)\cos \alpha_m x, \quad (4.11)$$

$$p_{zj}^{(2)}(x,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(T_{jmn}\cos n\phi + T'_{jmn}\sin n\phi)\cos \alpha_m x, \quad (4.12)$$

$$p_{\phi j}(x,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(Y_{jmn}\sin n\phi + Y'_{jmn}\cos n\phi)\cos \alpha_m x, \quad (4.13)$$

$$p_{xj}(x,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(X_{jmn}\cos n\phi + X'_{jmn}\sin n\phi)\sin \alpha_m x, \quad (4.14)$$

where the symmetric Fourier coefficients (Z_{jmn}, T_{jmn}, Y_{jmn}, X_{jmn}) are given by:

$$Z_{jmn} = -\left(\frac{1}{\pi a}\right)\left(\frac{2}{\ell}\right)\int_0^\ell Z_j(x)\cos\alpha_m x\,dx), \tag{4.15}$$

$$Z_{jmn} = -\left(\frac{1}{\pi n\varepsilon_{1j}}\right)\cos n\phi_j\sin\frac{n\varepsilon_{1j}}{a}\left(\frac{2}{\ell}\int_0^\ell Z_j(x)\cos\alpha_m x\,dx\right), \tag{4.16}$$

$$T_{jmo} = 0$$

$$T_{jmn} = \frac{2}{\pi n\varepsilon_{2j}}\sin(n\phi_j)\left\{1-\cos\frac{n\varepsilon_{2j}}{a}\right\}\left(\frac{2}{\ell}\int_0^\ell Q_j(x)\cos\alpha_m x\,dx\right), \tag{4.17}$$

$$Y_{jmo} = 0,$$

$$Y_{jmn} = \left(\frac{1}{\pi n\varepsilon_{3j}}\right)\sin(n\phi_j)\sin\frac{n\varepsilon_{3j}}{a}\left(\frac{2}{\ell}\int_0^\ell Y_j(x)\cos\alpha_m x\,dx\right), \tag{4.18}$$

$$X_{jmo} = -\left(\frac{1}{\pi a}\right)\left(\frac{2}{\ell}\int_0^\ell X_j(x)\sin\alpha_m x\,dx\right), \tag{4.19}$$

$$X_{jmn} = -\left(\frac{1}{\pi n\varepsilon_{4j}}\right)\cos(n\phi_j)\sin\frac{n\varepsilon_{4j}}{a}\left(\frac{2}{\ell}\int_0^\ell X_j(x)\sin\alpha_m x\,dx\right), \tag{4.20}$$

and the antisymmetric Fourier coefficients (Z'_{jmn}, T'_{jmn}, Y'_{jmn}, X'_{jmn}) are given by:

$$Z'_{jmo} = 0, \tag{4.21}$$

$$Z'_{jmn} = -\left(\frac{1}{\pi n \varepsilon_{1j}}\right)\sin(n\phi_j)\sin\frac{n\varepsilon_{1j}}{a}\left(\frac{2}{\ell}\int_0^\ell Z_j(x)\cos\alpha_m x\,dx\right), \tag{4.22}$$

$$T'_{jmo} = 0, \tag{4.23}$$

$$T'_{jmn} = -\left(\frac{2}{\pi n \varepsilon_{2j}}\right)\cos(n\phi_j)\left(1-\cos\frac{n\varepsilon_{2j}}{a}\right)\left(\frac{2}{\ell}\int_0^\ell Q_j(x)\cos\alpha_m x\,dx\right), \tag{4.24}$$

$$Y'_{jmo} = -\left(\frac{1}{\pi a}\right)\left(\frac{2}{\ell}\int_0^\ell Y_{jx}(x)\cos\alpha_m x\,dx\right), \tag{4.25}$$

$$Y'_{jmn} = -\left(\frac{1}{\pi n \varepsilon_{3j}}\right)\cos(n\phi_j)\sin\frac{n\varepsilon_{3j}}{a}\left(\frac{2}{\ell}\int_0^\ell Y_j(x)\cos\alpha_m x\,dx\right), \tag{4.26}$$

$$X'_{jmo} = 0, \tag{4.27}$$

$$X'_{jmn} = -\left(\frac{1}{\pi n \varepsilon_{4j}}\right)\sin(n\phi_j)\sin\frac{n\varepsilon_{4j}}{a}\left(\frac{2}{\ell}\int_0^\ell X_j(x)\sin\alpha_m x\,dx\right). \tag{4.28}$$

It is assumed that the resultant forces $X_j(x)$, $Y_j(x)$, $Z_j(x)$ and the resultant moment $T_j(x) = Q_j(x)\varepsilon_{2j}$ acting on the jth stringer may be expanded into Fourier series as odd or even functions with a period of 2ℓ as follows:

$$X_j(x) = \sum_{m=1}^{\infty} X_{jm}\sin\alpha_m x, \tag{4.29}$$

$$Y_j(x) = \sum_{m=0}^{\infty} Y_{jm} \cos \alpha_m x, \tag{4.30}$$

$$Z_j(x) = \sum_{m=0}^{\infty} Z_{jm} \cos \alpha_m x, \tag{4.31}$$

$$T_j(x) = \sum_{m=0}^{\infty} T_{jm} \cos \alpha_m x, \tag{4.32}$$

where

$$X_{jo} = 0, \tag{4.33}$$

$$X_{jm} = \frac{2}{\ell} \int_0^{\ell} X_j(x) \sin \alpha_m x \, dx, \tag{4.34}$$

$$Y_{jo} = \frac{1}{\ell} \int_0^{\ell} Y_j(x) dx, \tag{4.35}$$

$$Y_{jm} = \frac{2}{\ell} \int_0^{\ell} Y_j(x) \cos \alpha_m x \, dx, \tag{4.36}$$

$$Z_{jo} = \frac{1}{\ell} \int_0^{\ell} Z_j(x) dx, \tag{4.37}$$

$$Z_{jm} = \frac{2}{\ell} \int_0^{\ell} Z_j(x) \cos \alpha_m x \, dx, \tag{4.38}$$

$$T_{jo} = \frac{1}{\ell} \int_0^{\ell} T_j(x) dx, \tag{4.39}$$

$$T_{jm} = \frac{2}{\ell} \int_0^{\ell} T_j(x) \cos \alpha_m x \, dx. \tag{4.40}$$

Inserting Equations (4.33) through (4.40) into (4.15) through (4.28) gives:

$$Z_{jmn} = f_1(n)\cos(n\phi_j)Z_{jm}, \tag{4.41}$$

$$T_{jmn} = f_2(n)\sin(n\phi_j)T_{jm}, \tag{4.42}$$

$$Y_{jmn} = f_3(n)\sin(n\phi_j)Y_{jm}, \tag{4.43}$$

$$X_{jmn} = f_4(n)\cos(n\phi_j)X_{jm}, \tag{4.44}$$

$$Z'_{jmn} = f_1(n)\sin(n\phi_j)Z_{jm}, \tag{4.45}$$

$$T'_{jmn} = f_2(n)\cos(n\phi_j)T_{jm}, \tag{4.46}$$

$$Y'_{jmn} = f_3(n)\cos(n\phi_j)Y_{jm}, \tag{4.47}$$

$$X'_{jmn} = f_4(n)\sin(n\phi_j)X_{jm}, \tag{4.48}$$

where the following notation has been introduced:

$$f_1(n) = -\frac{1}{\pi n\varepsilon_{1j}}\sin\frac{n}{a}\varepsilon_{ij}, \tag{4.49}$$

$$f_2(n) = \frac{2}{\pi n\varepsilon_{2j}}\left[1 - \cos\frac{n}{a}\varepsilon_{2j}\right], \tag{4.50}$$

$$f_3(n) = -\frac{1}{\pi n \varepsilon_{3j}} \sin\frac{n}{a}\varepsilon_{3j}, \tag{4.51}$$

$$f_4(n) = -\frac{1}{\pi n \varepsilon_{4j}} \sin\frac{n}{a}\varepsilon_{4j}. \tag{4.52}$$

Therefore, the total surface loading on the cylinder is given by:

$$p_x(x,\phi) = \sum_{j=1}^{K}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(X_{jmn}\cos n\phi + X'_{jmn}\sin n\phi)\sin\alpha_m x, \tag{4.53}$$

$$p_\phi(x,\phi) = \sum_{j=1}^{K}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(Y_{jmn}\sin n\phi + Y'_{jmn}\cos n\phi)\cos\alpha_m x, \tag{4.54}$$

$$p_z(x,\phi) = \sum_{j=1}^{K}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}([Z_{jmn} + T_{jmn}]\cos n\phi + [Z'_{jmn} + T'_{jmn}]\sin n\phi)$$

$$\cos\alpha_m x \tag{4.55}$$

where K is the total number of stringers attached to the shell.

Solution of Donnell's Equations

In considering a circular cylindrical shell under distributed loadings, one has to solve the complete, inhomogenous Donnell equations (as given by (4.4), (4.5), and (4.6)) including the in-plane inertia terms. To solve these equations, the method of Fourier series was employed. In doing this, solutions are obtained which can be represented as follows:

$$U = U_c + U_p, \tag{4.56}$$

$$V = V_c + V_p, \tag{4.57}$$

$$W = W_c + W_p, \tag{4.58}$$

where the subscripts c and p refer to the complementary and particular solutions, respectively. The particular solution is used to account for the distributed load and the complementary solution is required in order that the combination of the two solutions will satisfy the boundary conditions of the problem. Furthermore, for convenience the symmetric and antisymmetric parts of the complementary and particular solutions are treated separately. Then since the partial differential equations are linear, the general solution is recovered by superposing the symmetric and antisymmetric parts. If it happens that the particular solution also satisfies the boundary conditions, then it is not necessary to determine the complementary solution. For free vibration, we seek solutions for the middle displacements (U,V,W) of the shell in the form:

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \sin \omega t \tag{4.59}$$

where ω denotes the natural frequency of the composite structure.

According to Flügge [8], the symmetric part of the complementary solution of Equations (4.4), (4.5), and (4.6) can be written in the following form:

$$u_c^s(x,\phi,t) = \{ \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} A_{sn} e^{\lambda_s \frac{x}{a}} \cos n\phi \} \sin \omega t \qquad (4.60)$$

$$v_c^s(x,\phi,t) = \{ \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} B_{sn} e^{\lambda_s \frac{x}{a}} \sin n\phi \} \sin \omega t \qquad (4.61)$$

$$w_c^s(x,\phi,t) = \{ \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} C_{sn} e^{\lambda_s \frac{x}{a}} \cos n\phi \} \sin \omega t \qquad (4.62)$$

From (4.60), (4.61), and (4.62), we see that each displacement is a product of two functions, each of which is a function of one of the variables x or ϕ. The independent variable ϕ represents a periodic motion with n waves around the circumference of the cylinder. Inserting (4.60), (4.61), and (4.62) into (4.4), (4.5), and (4.6) and setting $P_x = P_\phi = P_z = 0$ (complementary solution) results in a set of linear homogenous equations in the constants A_{sn}, B_{sn}, and C_{sn} as follows:

$$A_{sn}\left[\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)}{(1-\nu)}\,\Omega n^2\left(1 - \frac{\lambda_s^2}{n^2}\right) + n^4\left(1 - \frac{\lambda_s^2}{n^2}\right)^2\right] =$$

$$C_{sn}\lambda_s\left[\frac{2\nu\Omega}{1-\nu} + n^2\left(1 + \nu\,\frac{\lambda_s^2}{n^2}\right)\right], \qquad (4.63)$$

$$s=1,\ldots,8$$

$$B_{sn}\left[\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)}{(1-\nu)}\,\Omega n^2\left(1 - \frac{\lambda_s^2}{n^2}\right) + n^4\left(1 - \frac{\lambda_s^2}{n^2}\right)\right] =$$

$$C_{sn}(n)\left[\frac{-2\Omega}{1-\nu} + n^2\left(1 - (2+\nu)\,\frac{\lambda_s^2}{n^2}\right)\right], \quad (4.64)$$

$$s=1,\ldots,8$$

and,

$$(1-\nu)(1-\nu^2)\lambda_s^4 = 2\Omega^3 - \Omega^2[2+(3-\nu)n^2(1-\lambda_s^2/n^2) + 2kn^4(1-\lambda_s^2/n^2)^2] +$$

$$\Omega[(3-\nu)n^2(1-\lambda_s^2/n^2) - 2n^2(1-\nu^2\lambda_s^2/n^2) +$$

$$(1-\nu)n^4(1-\lambda_s^2/n^2) + (3-\nu)kn^6(1-\lambda_s^2/n^2)^3] -$$

$$(1-\nu)kn^8(1-\lambda_s^2/n^2)^4 \qquad s=1,\ldots,8 \qquad (4.65)$$

where

$$\Omega = (1-\nu^2)\rho a^2\omega^2/E \quad \text{and} \quad k = h^2/12a^2.$$

Equation (4.65) can be written as follows:

$$\lambda_s^8 + \lambda_s^6\left[-4n^2 + \Omega\left(\frac{3-\nu}{1-\nu}\right)\right] \quad \lambda_s^4\left[\frac{1-\nu}{k} + \frac{2\Omega^2}{(1-\nu)} - \Omega/k - 3\Omega_n^2(3-\nu)/(1-\nu) + 6n^4\right] +$$

$$\lambda_s^2[-4n^6 + 3\Omega n^4(3-\nu)/(1-\nu) + 2\Omega n^2/k - 2\Omega\nu^2/k(1-\nu) +$$

$$\Omega(3-\nu)/k(1-\nu) - 4n^2\Omega^2/(1-\nu) - \Omega^2(3-\nu)/k(1-\nu)] +$$

$$[-2\Omega^3/k(1-\nu) + 2\Omega^2/k(1-\nu) + \Omega^2 n^2(3-\nu)/k(1-\nu) + 2n^4\Omega^2/(1-\nu) -$$

$$\Omega n^2(3-\nu)/k(1-\nu) + 2n^2\Omega/k(1-\nu) - \Omega_n^4/k -$$

$$\Omega_n^6(3-\nu)/(1-\nu) + n^8] = 0 \tag{4.66}$$

The complementary solution for the antisymmetric part of Equations (4.4), (4.5) and (4.6) can be written in the following form:

$$u_c^a(x,\phi,t) = \{\sum_{n=1}^{\infty}\sum_{s=1}^{8} A'_{sn} e^{\lambda_s \frac{x}{a}} \sin n\phi\}\sin \omega t, \tag{4.67}$$

$$v_c^a(x,\phi,t) = -\{\sum_{n=1}^{\infty}\sum_{s=1}^{8} B'_{sn} e^{\lambda_s \frac{x}{a}} \cos n\phi\}\sin \omega t, \tag{4.68}$$

and

$$w_c^a(x,\phi,t) = \{\sum_{n=1}^{\infty}\sum_{s=1}^{8} C'_{sn} e^{\lambda_s \frac{x}{a}} \sin n\phi\}\sin \omega t \tag{4.69}$$

where the superscript a represents the antisymmetric part of the solution. Inserting (4.67), (4.68), and (4.69) into (4.4), (4.5), and (4.6) and letting $p_x = p_\phi = p_z = 0$ result in a set of linear homogenous equations in the constants A'_{sn}, B'_{sn}, and C'_{sn} identical with (4.63), (4.64), and (4.65). In general the roots of Equation (4.66) will have the form:

$$\lambda_s = (\psi_1 \pm i\theta_1),\ (\psi_2 \pm i\theta_2),\ (\psi_3 \pm i\theta_3),\ (\psi_4 \pm i\theta_4) \tag{4.70}$$

where ψ_1, ψ_2, ψ_3, ψ_4, θ_1, θ_2, θ_3, and θ_4 are real quantities.

The complementary solution corresponding to the roots of (4.70) may be written as follows:

$$\begin{aligned} u_c^s(x,\phi,t) = \{ \sum_{n=0}^{\infty} \{ & e^{\psi_{1n} \frac{x}{a}} \left[(A_1+A_2)\cos \theta_{1n} \frac{x}{a} + (A_1-A_2) i \sin \theta_{1n} \frac{x}{a} \right] + \\ & e^{\psi_{2n} \frac{x}{a}} \left[(A_3+A_4)\cos \theta_{2n} \frac{x}{a} + (A_3-A_4/i \sin \theta_{2n} \frac{x}{a} \right] + \\ & e^{\psi_{3n} \frac{x}{a}} \left[(A_5+A_6)\cos \theta_{3n} \frac{x}{a} + (A_5-A_6) i \sin \theta_{3n} \frac{x}{a} \right] + \\ & e^{\psi_{4n} \frac{x}{a}} \left[(A_7+A_8)\cos \theta_{4n} \frac{x}{a} + (A_7-A_8) i \sin \theta_{4n} \frac{x}{a} \right] \}\cos n\phi \} \\ & \sin \omega t, \end{aligned} \tag{4.71}$$

$$v_c^s(x,\phi,t) = \{\sum_{n=1}^{\infty} \{e^{\psi_{1n}\frac{x}{a}}\left[(B_1+B_2)\cos\theta_{1n}\frac{x}{a} + (B_1-B_2)i\sin\theta_{1n}\frac{x}{a}\right]$$

$$+ e^{\psi_{2n}\frac{x}{a}}\left[(B_3+B_4)\cos\theta_{2n}\frac{x}{a} + (B_3-B_4)i\sin\theta_{2n}\frac{x}{a}\right]$$

$$+ e^{\psi_{3n}\frac{x}{a}}\left[(B_5+B_6)\cos\theta_{3n}\frac{x}{a} + (B_5-B_6)i\sin\theta_{3n}\frac{x}{a}\right]$$

$$+ e^{\psi_{4n}\frac{x}{a}}\left[(B_7+B_8)\cos\theta_{4n}\frac{x}{a} + \right.$$

$$\left. + (B_7-B_8)i\sin\theta_{4n}\frac{x}{a}\right]\}\sin n\phi\}\sin\omega t, \qquad (4.72)$$

$$w_c^s(x,\phi,t) = \{\sum_{n=0}^{\infty} \{e^{\psi_{1n}\frac{x}{a}}\left[(C_1+C_2)\cos\theta_{1n}\frac{x}{a} + (C_1-C_2)i\sin\theta_{1n}\frac{x}{a}\right]$$

$$+ e^{\psi_{2n}\frac{x}{a}}\left[(C_3+C_4)\cos\theta_{2n}\frac{x}{a} + (C_3-C_4)i\sin\theta_{2n}\frac{x}{a}\right]$$

$$+ e^{\psi_{3n}\frac{x}{a}}\left[(C_5+C_6)\cos\theta_{3n}\frac{x}{a} + (C_5-C_6)i\sin\theta_{3n}\frac{x}{a}\right]$$

$$+ e^{\psi_{4n}\frac{x}{a}}\left[(C_7+C_8)\cos\theta_{4n}\frac{x}{a} + \right.$$

$$\left. + (C_7-C_8)i\sin\theta_{4n}\frac{x}{a}\right]\}\cos n\phi\}\sin\omega t \qquad (4.73)$$

Examination of Equation (4.73) shows that it is convenient to define a new set of arbitrary constants as follows:

$$\bar{C}_1 = C_1 + C_2,$$

$$\bar{C}_2 = (C_1 - C_2)i,$$

$$\bar{C}_3 = C_3 + C_4,$$

$$\bar{C}_4 = (C_3 - C_4)i,$$

$$\bar{C}_5 = C_5 + C_6,$$

$$\bar{C}_6 = (C_5 - C_6)i,$$

$$\bar{C}_7 = C_7 + C_8,$$

$$\bar{C}_8 = (C_7 - C_8)i. \tag{4.74}$$

The constants A_i and B_i can be determined in terms of the C_i for each integer n. Examination of (4.63) reveals that it may be written as follows:

$$A_i = \alpha_i C_i \tag{4.75}$$

where

$$\alpha_1 = \bar{\alpha}_1 + i\bar{\alpha}_2,$$

$$\alpha_2 = \bar{\alpha}_1 + i\bar{\alpha}_2$$

$$\alpha_3 = \bar{\alpha}_3 + i\bar{\alpha}_4$$

$$\alpha_4 = \bar{\alpha}_3 - i\bar{\alpha}_4$$

$$\alpha_5 = \bar{\alpha}_5 - i\bar{\alpha}_6$$

$$\alpha_6 = \bar{\alpha}_5 - i\bar{\alpha}_6$$

$$\alpha_7 = \bar{\alpha}_7 + i\bar{\alpha}_8$$

$$\alpha_8 = \bar{\alpha}_7 - i\bar{\alpha}_8 \qquad (4.76)$$

By employing Equations (4.74), (4.75), and (4.76) the arbitrary coefficients of (4.71) can be written in the following manner:

$$A_1 + A_2 = \bar{\alpha}_1\bar{C}_1 + \bar{\alpha}_2\bar{C}_2,$$

$$A_3 + A_4 = \bar{\alpha}_3\bar{C}_3 + \bar{\alpha}_4\bar{C}_4,$$

$$A_5 + A_6 = \bar{\alpha}_5\bar{C}_5 + \bar{\alpha}_6\bar{C}_6,$$

$$A_7 + A_8 = \bar{\alpha}_7\bar{C}_7 + \bar{\alpha}_8\bar{C}_8,$$

$$(A_1-A_2)i = -\bar{\alpha}_2\bar{C}_1 + \bar{\alpha}_1\bar{C}_2,$$

$$(A_3-A_4)i = -\bar{\alpha}_4\bar{C}_3 + \bar{\alpha}_3\bar{C}_4,$$

$$(A_5-A_6)i = -\bar{\alpha}_6\bar{C}_5 + \bar{\alpha}_5\bar{C}_6,$$

and

$$(A_7-A_8)i = -\bar{\alpha}_8\bar{C}_7 + \bar{\alpha}_7\bar{C}_8. \tag{4.77}$$

Also, (4.64) has the form:

$$B_i = \beta_i C_i \tag{4.78}$$

where

$$\beta_1 = \bar{\beta}_1 + i\bar{\beta}_2,$$

$$\beta_2 = \bar{\beta}_1 - i\bar{\beta}_2,$$

$$\beta_3 = \bar{\beta}_3 + i\bar{\beta}_4,$$

$$\beta_4 = \bar{\beta}_3 - i\bar{\beta}_4,$$

$$\beta_5 = \bar{\beta}_5 + i\bar{\beta}_6,$$

$$\beta_6 = \bar{\beta}_5 - i\bar{\beta}_6,$$

$$\beta_7 = \bar{\beta}_7 + i\bar{\beta}_8,$$

and

$$\beta_8 = \bar{\beta}_7 - i\bar{\beta}_8. \tag{4.79}$$

With the help of (4.74), (4.78), and (4.79), the arbitrary coefficients of (4.72) can be shown to be:

$$B_1 + B_2 = \bar{\beta}_1\bar{C}_1 + \bar{\beta}_2\bar{C}_2$$

$$B_3 + B_4 = \bar{\beta}_3\bar{C}_3 + \bar{\beta}_4\bar{C}_4$$

$$B_5 + B_6 = \bar{\beta}_5\bar{C}_5 + \bar{\beta}_6\bar{C}_6$$

$$B_7 + B_8 = \bar{\beta}_7\bar{C}_7 + \bar{\beta}_8\bar{C}_8$$

$$(B_1 - B_2)i = \bar{\beta}_1\bar{C}_2 + \bar{\beta}_2\bar{C}_1$$

$$(B_3 - B_4)i = \bar{\beta}_3\bar{C}_4 - \bar{\beta}_4\bar{C}_3$$

$$(B_5 - B_6)i = \bar{\beta}_5\bar{C}_6 - \bar{\beta}_6\bar{C}_5$$

$$(B_7 - B_8)i = \bar{\beta}_7\bar{C}_8 - \bar{\beta}_8\bar{C}_7 \tag{4.80}$$

Inserting Equations (4.77) and (4.80) into Equations (4.71), (4.72), and (4.73) yields the final form of the solution in terms of the new set of eight arbitrary constants $\bar{C}_i$ and the real terms $\bar{\alpha}_i$ and $\bar{\beta}_i$ as follows:

$$u_c^s = \{\sum_{n=0}^{\infty} \{e^{\psi_{1n}\frac{x}{a}}\left[(\bar{\alpha}_{1n}\bar{C}_{1n} + \bar{\alpha}_{2n}\bar{C}_{2n})\cos\theta_{1n}\frac{x}{a} + \right.$$

$$\left. + (\bar{\alpha}_{1n}\bar{C}_{2n} - \bar{\alpha}_{2n}\bar{C}_{1n})\sin\theta_{1n}\frac{x}{a}\right]$$

$$+ e^{\psi_{2n}\frac{x}{a}}\left[(\bar{\alpha}_{3n}\bar{C}_{3n} + \bar{\alpha}_{4n}\bar{C}_{4n})\cos\theta_{2n}\frac{x}{a} + (\bar{\alpha}_{3n}\bar{C}_{4n} - \bar{\alpha}_{4n}\bar{C}_{3n})\sin\theta_{2n}\frac{x}{a}\right]$$

$$+ e^{\psi_{3n}\frac{x}{a}}\left[(\bar{\alpha}_{5n}\bar{C}_{5n} + \bar{\alpha}_{6n}\bar{C}_{6n})\cos\theta_{3n}\frac{x}{a} + (\bar{\alpha}_{5n}\bar{C}_{6n} - \bar{\alpha}_{6n}\bar{C}_{5n})\sin\theta_{3n}\frac{x}{a}\right]$$

$$+ e^{\psi_{4n}\frac{x}{a}}\left[(\bar{\alpha}_{7n}\bar{C}_{7n} + \bar{\alpha}_{8n}\bar{C}_{8n})\cos\theta_{4n}\frac{x}{a} + \right.$$

$$\left. + (\bar{\alpha}_{7n}\bar{C}_{8n} - \bar{\alpha}_{8n}\bar{C}_{7n})\sin\theta_{4n}\frac{x}{a}\right]\}\cos n\theta\}\sin\omega t \tag{4.81}$$

$$v_c^s = \{\sum_{n=1}^{\infty} \{e^{\psi_{1n}\frac{x}{a}}\left[(\bar{\beta}_{1n}\bar{C}_{1n} + \bar{\beta}_{2n}\bar{C}_{2n})\cos\theta_{1n}\frac{x}{a} + \right.$$

$$\left. + (\bar{\beta}_{1n}\bar{C}_{2n} - \bar{\beta}_{2n}\bar{C}_{1n})\sin\theta_{1n}\frac{x}{a}\right]$$

$$+ e^{\psi_{2n}\frac{x}{a}}\left[\bar{\beta}_{3n}\bar{C}_{3n} + \bar{\beta}_{4n}\bar{C}_{4n})\cos\theta_{2n}\frac{x}{a} + (\bar{\beta}_{3n}\bar{C}_{4n} - \bar{\beta}_{4n}\bar{C}_{3n})\sin\theta_{2n}\frac{x}{a}\right]$$

$$+ e^{\psi_{3n}\frac{x}{a}}\left[(\bar{\beta}_{5n}\bar{C}_{5n} + \bar{\beta}_{6n}\bar{C}_{6n})\cos\theta_{3n}\frac{x}{a} + (\bar{\beta}_{5n}\bar{C}_{6n} - \bar{\beta}_{6n}\bar{C}_{5n})\sin\theta_{3n}\frac{x}{a}\right]$$

$$+ e^{\psi_{4n}\frac{x}{a}}\left[(\bar{\beta}_{7n}\bar{C}_{7n} + \bar{\beta}_{8n}\bar{C}_{8n})\cos\theta_{4n}\frac{x}{a} + \right.$$

$$\left. + (\bar{\beta}_{7n}\bar{C}_{8n} - \bar{\beta}_{8n}\bar{C}_{7n})\sin\theta_{4n}\frac{x}{a}\right]\}\sin n\phi\}\sin\omega t, \quad (4.82)$$

$$W_c^s = \{\sum_{n=0}^{\infty}\{e^{\psi_{1n}\frac{x}{a}}\left[\bar{C}_{1n}\cos\theta_{1n}\frac{x}{a} + \bar{C}_{2n}\sin\theta_{1n}\frac{x}{a}\right]$$

$$+ e^{\psi_{2n}\frac{x}{a}}\left[\bar{C}_{3n}\cos\theta_{2n}\frac{x}{a} + \bar{C}_{4n}\sin\theta_{2n}\frac{x}{a}\right]$$

$$+ e^{\psi_{3n}\frac{x}{a}}\left[\bar{C}_{5n}\cos\theta_{3n}\frac{x}{a} + \bar{C}_{6n}\sin\theta_{3n}\frac{x}{a}\right]$$

$$+ e^{\psi_{4n}\frac{x}{a}}\left[\bar{C}_{7n}\cos\theta_{4n}\frac{x}{a} + \bar{C}_{8n}\sin\theta_{4n}\frac{x}{a}\right]\}\cos n\phi\}\sin\omega t, \quad (4.83)$$

where the complex constants A_{sn}, B_{sn}, and C_{sn} have been replaced by the real terms $\bar{\alpha}_{in}$ and $\bar{\beta}_{in}$ and the real arbitrary constants $\bar{C}_{in}$, ψ_{1n}, ψ_{2n}, ψ_{3n}, and ψ_{4n} are the real parts and θ_{1n}, θ_{2n}, θ_{3n}, and θ_{4n} are the imaginary parts of the roots of Equation (4.66). The quantities $\bar{\alpha}_{in}$ and $\bar{\beta}_{in}$ are given in Appendix E.

The functions $e^{\psi_{1n}\frac{x}{a}}\cos\theta_{1n}\frac{x}{a}$, $e^{\psi_{1n}\frac{x}{a}}\sin\theta_{1n}\frac{x}{a}$, $e^{\psi_{2n}\frac{x}{a}}\cos\theta_{2n}\frac{x}{a}$, $e^{\psi_{3n}\frac{x}{a}}\cos\theta_{3n}\frac{x}{a}$, $e^{\psi_{3n}\frac{x}{a}}\sin\theta_{3n}\frac{x}{a}$, $e^{\psi_{4n}\frac{x}{a}}\cos\theta_{4n}\frac{x}{a}$, and $e^{\psi_{4n}\frac{x}{a}}\sin\theta_{4n}\frac{x}{a}$ which occur in the complementary solution satisfy Dirichlet's conditions for each integer n and are continuous in the closed interval $[0,\ell]$. Therefore, they can be expanded in uniformly convergent Fourier cosine series by extending them as even functions as follows:

$$e^{\psi_{1n}\frac{x}{a}} \cos \theta_{1n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{1mn} \cos \alpha_m x, \tag{4.84}$$

$$e^{\psi_{1n}\frac{x}{a}} \sin \theta_{1n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{2mn} \cos \alpha_m x, \tag{4.85}$$

$$e^{\psi_{2n}\frac{x}{a}} \cos \theta_{2n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{3mn} \cos \alpha_m x, \tag{4.86}$$

$$e^{\psi_{2n}\frac{x}{a}} \sin \theta_{2n} \frac{x}{a} + \sum_{m=0}^{\infty} t_{4mn} \cos \alpha_m x, \tag{4.87}$$

$$e^{\psi_{3n}\frac{x}{a}} \cos \theta_{3n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{5mn} \cos \alpha_m x, \tag{4.88}$$

$$e^{\psi_{3n}\frac{x}{a}} \sin \theta_{3n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{6mn} \cos \alpha_n x, \tag{4.89}$$

$$e^{\psi_{4n}\frac{x}{a}} \cos \theta_{4n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{7mn} \cos \alpha_m x, \tag{4.90}$$

$$e^{\psi_{4n}\frac{x}{a}} \sin \theta_{4n} \frac{x}{a} = \sum_{m=0}^{\infty} t_{8mn} \cos \alpha_m x, \tag{4.91}$$

where

$$t_{1mn} = \frac{1}{\ell}\left\{\frac{(-1)^m e^{\frac{\psi_{1n}\ell}{a}}\left[\frac{\psi_{1n}}{a}\cos\theta_{1n}\frac{\ell}{a} + \left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)\sin\theta_{1n}\frac{\ell}{a}\right] - \frac{\psi_{1n}}{a}}{\alpha_m^2 + \left[\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right]^2}\right. +$$

$$\left.\frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\cos\theta_{1n}\frac{\ell}{a} + \left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)\sin\theta_{1n}\frac{\ell}{a}\right] - \frac{\psi_{1n}}{a}}{\alpha_m^2 + \left[\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right]^2}\right\}, \quad (4.92)$$

$$t_{2mn} = \frac{1}{\ell}\left\{\frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\sin\theta_{1n}\frac{\ell}{a} - \left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)\cos\theta_{1n}\frac{\ell}{a}\right]}{\alpha_m^2 + \left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)^2}\right.$$

$$+ \frac{\left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)}{\alpha_m^2 + \left(\frac{\theta_{1n}}{a} + \frac{m\pi}{\ell}\right)^2} +$$

$$\left.\frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\sin\theta_{1n}\frac{\ell}{a} - \left(\frac{\theta_{1n}}{a} - \frac{m\pi}{\ell}\right)\cos\theta_{1n}\frac{\ell}{a}\right] + \left(\frac{\theta_{1n}}{a} - \frac{m\pi}{\ell}\right)}{\alpha_m^2 + \left(\frac{\theta_{1n}}{a} - \frac{m\pi}{\ell}\right)^2}\right\} \quad (4.93)$$

with similar expressions for the Fourier coefficients t_{3mn}, t_{4mn}, t_{5mn}, t_{6mn}, t_{7mn}, and t_{8mn} which can be obtained from t_{1mn} and t_{2mn} by by replacing ψ_{1n} and θ_{1n} with either ψ_{2n}, θ_{2n} or ψ_{3n}, θ_{3n} or ψ_{4n}, θ_{4n}.

The functions occurring in the complementary solution may also be expanded in uniformly convergent Fourier sine series as follows:

$$e^{\psi_{1n}\frac{x}{a}} \cos \theta_{1n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{1mn} \sin \alpha_m x, \tag{4.94}$$

$$e^{\psi_{1n}\frac{x}{a}} \sin \theta_{1n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{2mn} \sin \alpha_m x, \tag{4.95}$$

$$e^{\psi_{2n}\frac{x}{a}} \cos \theta_{2a} \frac{x}{a} = \sum_{m=1}^{\infty} r_{3mn} \sin \alpha_m x, \tag{4.96}$$

$$e^{\psi_{2n}\frac{x}{a}} \sin \theta_{2n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{4mn} \sin \alpha_m x, \tag{4.97}$$

$$e^{\psi_{3n}\frac{x}{a}} \cos \theta_{3n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{5mn} \sin \alpha_m x, \tag{4.98}$$

$$e^{\psi_{3n}\frac{x}{a}} \sin \theta_{3n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{6mn} \sin \alpha_m x, \tag{4.99}$$

$$e^{\psi_{4n}\frac{x}{a}} \cos \theta_{4n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{7mn} \sin \alpha_m x, \tag{4.100}$$

$$e^{\psi_{4n}\frac{x}{a}} \sin \theta_{4n} \frac{x}{a} = \sum_{m=1}^{\infty} r_{8mn} \sin \alpha_m x, \tag{4.101}$$

where

$$r_{1mn} = \frac{1}{\ell}\left\{\frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\sin\theta_{1n}\frac{\ell}{a} - \left(\frac{\theta_{1n}}{a}+\frac{m\pi}{\ell}\right)\cos\theta_{1n}\frac{\ell}{a}\right] + \left(\frac{\theta_{1n}}{a}+\frac{m\pi}{\ell}\right)}{\left(\frac{\psi_{1n}}{a}\right)^2 + \left(\frac{\theta_{1n}}{a}+\frac{m\pi}{\ell}\right)^2}\right.$$

$$\left. - \frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\sin\frac{\theta_{1n}\ell}{a} - \left(\frac{\theta_{1n}}{a}-\frac{m\pi}{\ell}\right)\cos\frac{\theta_{1n}\ell}{a}\right] + \left(\frac{\theta_{1n}}{a}-\frac{m\pi}{\ell}\right)}{\left(\frac{\psi_{1n}}{a}\right)^2 + \left(\frac{\theta_{1n}}{a}-\frac{m\pi}{\ell}\right)^2}\right\}, \tag{4.102}$$

$$r_{2mn} = \frac{1}{\ell}\left\{\frac{-(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\cos\frac{\theta_{1n}\ell}{a} + \left(\frac{\theta_{1n}}{a}+\frac{m\pi}{\ell}\right)\sin\frac{\theta_{1n}\ell}{a}\right] + \frac{\psi_{1n}\ell}{a}}{\left(\frac{\psi_{1n}}{a}\right)^2 + \left(\frac{\theta_{1n}}{a}+\frac{m\pi}{\ell}\right)^2} + \right.$$

$$\left. \frac{(-1)^m e^{\psi_{1n}\frac{\ell}{a}}\left[\frac{\psi_{1n}}{a}\cos\frac{\theta_{1n}\ell}{a} + \left(\frac{\theta_{1n}}{a}-\frac{m\pi}{\ell}\right)\sin\frac{\theta_{1n}\ell}{a}\right] - \psi_{1n}\frac{\ell}{a}}{\left(\frac{\psi_{1n}}{a}\right)^2 + \left(\frac{\theta_{1n}}{a}-\frac{m\pi}{\ell}\right)^2}\right\}. \tag{4.103}$$

with similar expressions for the Fourier coefficients r_{3mn}, r_{4mn}, r_{5mn}, r_{6mn}, r_{7mn}, and r_{8mn}.

By employing Equations (4.84) through (4.103), the symmetric part of the complementary solution may be written as:

$$u_c^s(x,\phi) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{1n}(\bar{\alpha}_{1n}r_{1mn} - \bar{\alpha}_{2n}r_{2mn})\sin\alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{2n}(\bar{\alpha}_{1n}r_{2mn} + \bar{\alpha}_{2n}r_{1mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{3n}(\bar{\alpha}_{3n}r_{3mn} - \bar{\alpha}_{4n}r_{4mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{4n}(\bar{\alpha}_{3n}r_{4mn} + \bar{\alpha}_{4n}r_{3mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{5n}(\bar{\alpha}_{5n}r_{5mn} - \bar{\alpha}_{6n}r_{6mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{6n}(\bar{\alpha}_{5n}r_{6mn} + \bar{\alpha}_{6n}r_{5mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{7n}(\bar{\alpha}_{7n}r_{7mn} - \bar{\alpha}_{8n}r_{8mn})\sin \alpha_m x \cos n\phi +$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \bar{C}_{8n}(\bar{\alpha}_{7n}r_{8mn} + \bar{\alpha}_{8n}r_{7mn})\sin \alpha_m x \cos n\phi, \quad (4.104)$$

$$v_c^s(x,\phi) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \bar{C}_{1n}(\bar{\beta}_{1n}t_{1mn} - \bar{\beta}_{2n}t_{2mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \bar{C}_{2n}(\bar{\beta}_{1n}t_{2mn} + \bar{\beta}_{2n}t_{1mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \bar{C}_{3n}(\bar{\beta}_{3n}t_{3mn} - \bar{\beta}_{4n}t_{4mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{4n}(\bar{\beta}_{3n}t_{4mn} + \bar{\beta}_{4n}t_{3mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{sn}(\bar{\beta}_{5n}t_{5mn} - \bar{\beta}_{6n}t_{6mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{6n}(\bar{\beta}_{5n}t_{6mn} + \bar{\beta}_{6n}t_{5mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{7n}(\bar{\beta}_{7n}t_{7mn} - \bar{\beta}_{8n}t_{8mn})\cos \alpha_m x \sin n\phi +$$

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{8n}(\bar{\beta}_{7n}t_{8mn} + \bar{\beta}_{8n}t_{7mn})\cos \alpha_m x \sin n\phi, \tag{4.105}$$

$$w_c^s(x,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{1n}t_{1mn}\cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{2n}t_{2mn}\cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{3n}t_{3mn}\cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{4n}t_{4mn}\cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{5n}t_{5mn}\cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{6n} t_{6mn} \cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{7n} t_{7mn} \cos \alpha_m x \cos n\phi +$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{C}_{8n} t_{8mn} \cos \alpha_m x \cos n\phi. \tag{4.106}$$

The particular solutions of Equations (4.4), (4.5), and (4.6) corresponding to the distributed loadings is given by (4.11), (4.12), (4.13), and (4.14) for the symmetric case are given by:

$$u_p^s = \sum_{j=1}^{K} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \{g_5 Z_{jmn} + g_5 T_{jmn} + g_6 Y_{jmn} + g_a X_{jmn}\} \sin \alpha_m x \cos n\phi, \tag{4.107}$$

$$v_p^s = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{g_3 Z_{jmn} + g_3 T_{jmn} + g_4 Y_{jmn} + g_8 X_{jmn}\} \cos \alpha_m x \sin n\phi, \tag{4.108}$$

$$w_p^s = \sum_{j=1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{g_1 Z_{jmn} + g_1 T_{jmn} + g_2 Y_{jmn} + g_7 X_{jmn}\} \cos \alpha_m x \cos n\phi, \tag{4.109}$$

where g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, and g_9 are functions of m, n, ρ, ν, E, h, a, and ω^2 given in Appendix F.

Therefore, the symmetric part of the general solution of (4.4), (4.5), and (4.6) is given by:

$$u^s = u_c^s + u_p^s, \tag{4.110}$$

$$v^s = v_c^s + v_p^s, \tag{4.111}$$

$$w^s = w_c^s + w_p^s. \tag{4.112}$$

The antisymmetric part of the complementary solution of Donnell's equations may be readily obtained by the replacement of sin $n\phi$ for cos $n\phi$ and -cos $n\phi$ for sin $n\phi$ and replacing $\bar{C}_{sn}$ by $\bar{C}'_{sn}$ in (4.104), (4.105), and (4.106). This solution is denoted by u_c^a, v_c^a, and w_c^a. Similarly, the particular solution of (4.4), (4.5), and (4.6) for the antisymmetric part of the loading can be obtained by the replacement of sin $n\phi$ for cos $n\phi$ and -cos $n\phi$ for sin $n\phi$ in (4.107), (4.108), and (4.109). This solution is denoted by u_p^a, v_p^a, and w_p^a.

Therefore, the antisymmetric part of the general solution of (4.4), (4.5), and (4.6) is given by:

$$u^a = u_c^a + u_p^a, \tag{4.113}$$

$$v^a = v_c^a + v_p^a, \tag{4.114}$$

and

$$w^a = w_c^a + w_p^a, \tag{4.115}$$

and the general solution obtained by superposing solutions is given by:

$$u = u^s + u^a, \tag{4.116}$$

$$v = v^s + v^a, \tag{4.117}$$

and

$$w = w^s + w^a. \tag{4.118}$$

Equations of Motion for the Stringers

The assumptions made concerning the stringers are the same as those given in Appendix A. V. Z. Vlasov's equations for extension, bending, and torsional vibrations of a thin-walled beam neglecting rotatory inertia (see Figure 11) are:

$$E_j^* A_j \frac{d^2 u_{oj}}{dx^2} + \rho_j A_j \omega^2 u_{oj} = X_j, \tag{4.119}$$

$$E_j^* I_{z'_j} \frac{d^4 v_{oj}}{dx^4} + E_j^* I_{y'_j z'_j} \frac{d^4 w_{oj}}{dx^4} + \rho_j I_{z'_j} \omega^2 \frac{d^2 v_{oj}}{dx^2} -$$

$$- \rho_j A_j \omega^2 v_{oj} + \rho_j A_j a_{z'_j} \omega^2 \theta_{oj} = Y_j, \tag{4.120}$$

$$E_j^* I_{y'_j} \frac{d^4 w_{oj}}{dx^4} + E_j^* I_{y'_j z'_j} \frac{d^4 v_{oj}}{dx^4} + \rho_j I_{y'_j} \omega^2 \frac{d^2 w_{oj}}{dx^2} -$$

$$- \rho_j A_j \omega^2 w_{oj} - \rho_j A_j a_{y_j} \omega^2 \theta_{oj} = Z_j, \tag{4.121}$$

and

$$E_j^* C_{wj} \frac{d^4\theta_{oj}}{dx^4} - (GJ)_j \frac{d^2\theta_{oj}}{dx^2} + \rho_j A_j \omega^2 a_{zj} v_{oj} -$$

$$- \rho_j A_j a_{yj} \omega^2 w_{oj} - \rho_j I_{oj} \omega^2 \theta_{oj} = T_j + z_j Y_j, \qquad (4.122)$$

where the following notation has been employed:

u_{oj} = deflection of shear center in the x-direction,

v_{oj} = deflection of shear center in the ϕ-direction,

w_{oj} = deflection of shear center in the z-direction,

θ_{oj} = rotation of the cross section,

$I_{y'z'}, I_{z'j}$ = principal centroidal moments of inertia,

A_j = cross sectional area,

J = St. Venant constant for uniform torsion,

ρ_j = mass density,

I_{oj} = polar moment of inertia about the centroid,

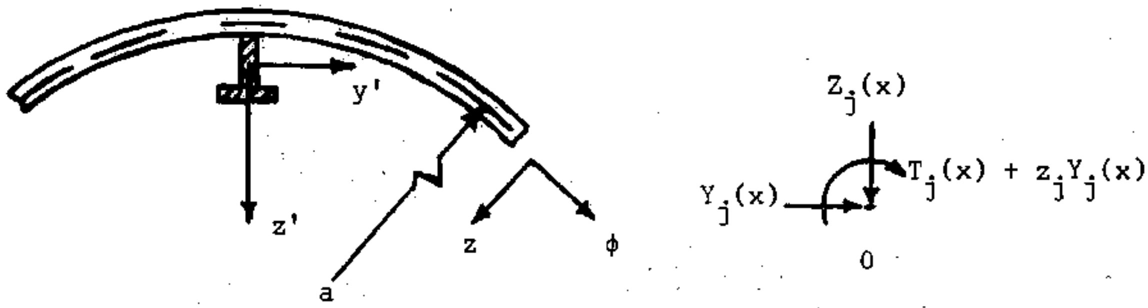
j = subscript indicating the stringer at location,

X_j, Y_j = resultant longitudinal and tangential forces transmitted from the shell to the stringer,

Z_j = resultant radial force,

T_j = resultant torque,

z_j = perpendicular distance from attachment line to shear center.



(a) Geometry of Stiffened Cylinder

(b) Forces and Moments Transmitted from Shell to Stringer

Figure 12. Shell and Stringer Coordinate Systems and Loading on Stringers

The first of V. Z. Vlasov's equations determines the longitudinal vibrations of the thin-walled rib and is not coupled to the other equations. The remaining three equations constitute a coupled system of ordinary linear differential equations with constant coefficients for the bending-torsional response of the stringers. Since each equation is of fourth order, their complementary solution will contain twelve arbitrary constants. Since the characteristic equation for this system will be a polynomial of degree 12, it is not possible to find a closed form expression for the solution. Furthermore, the precise form of the complementary solutions to these equations depends upon the relative magnitudes of the cross sectional constants.

For stringers with certain geometric properties, it is possible to uncouple the stringer equations of motion and thereby obtain a closed form solution. The three cases which fall into this category are given by

(1) the shear center of the stringer coincides with the centroid and the stringer has two axes of symmetry ($I_{y'z'} = 0$),

(2) the shear center lies on a vertical line above or below the centroid ($a_y = 0$, $a_z \pm 0$) and has a single axis of symmetry ($I_{y'z'} = 0$), and

(3) the shear center coincides with the centroid but the stringer has a nonsymmetrical cross section ($I_{yz} \neq 0$).

For the sake of light weight construction and ease of fabrication, the stringers used in the aircraft industry are generally thin-walled members of which the () rectangular-shape, () hat-shape, and ()-shape are frequent examples. These examples illustrate cases 1, 2, and 3, respectively. Therefore, these cases are technically important as well as of academic interest and are now discussed in some detail.

For the stringer with two axes of symmetry, the equations of motion for the bending-torsional response will be uncoupled. Inserting Equations (4.29), (4.30), (4.31), and (4.32) into Equations (4.119), (4.120), (4.121), and (4.122), and letting ($a_y = a_z = I_{yz} = 0$ yields the following set of equations:

$$E_j^* A_j \frac{d^2 u_{oj}}{dx^2} + \rho_j A_j \omega^2 u_{oj} = \sum_{m=0}^{\infty} X_{jm} \sin \frac{m\pi x}{\ell} , \tag{4.123}$$

$$E_j^* I_{zj} \frac{d^4 v_{oj}}{dx^4} + \rho_j I_{zj} \frac{d^2 v_{oj}}{dx^2} \omega^2 - \rho_j A_j \omega^2 v_{oj} =$$

$$= \sum_{m=0}^{\infty} Y_{jm} \cos \frac{m\pi x}{\ell} , \tag{4.124}$$

$$E_j^* I_{yj} \frac{d^4 w_{oj}}{dx^4} + \rho_j I_{yj} \frac{d^2 w_{oj}}{dx^2} \omega^2 - \rho_j A_j \omega^2 w_{oj} = \sum_{m=0}^{\infty} Z_{jm} \cos \frac{m\pi x}{\ell} \quad (4.125)$$

$$E_y^* C_{wj} \frac{d^4 \theta_{oj}}{dx^4} - (GJ)_j \frac{d^2 \theta_{oj}}{dx^2} - \rho_j I_{oj} \omega^2 \theta_{oj} =$$

$$\sum_{m=0}^{\infty} T_{jm} \cos \frac{m\pi x}{\ell} + z_j \sum_{m=0}^{\infty} Y_{jm} \cos \frac{m\pi x}{\ell} \quad (4.126)$$

The general solution of these equations is given by

$$u_{oj}(x) = U_{ij} \cos \gamma_{4j} x + U_{2j} \sin \gamma_{4j} x + \sum_{m=0}^{\infty} \zeta_{1m} X_{jm} \sin \frac{m\pi x}{\ell} \quad (4.127)$$

$$v_{oj}(x) = D_{1j} e^{\mu_{1j} x} + D_{2j} e^{-\mu_{1j} x} + D_{3j} \cos \mu_2 x + D_{4j} \sin \mu_2 x +$$

$$+ \sum_{m=0}^{\infty} \bar{\zeta}_{om} Y_{jm} \cos \alpha_m x \quad (4.128)$$

$$w_{oj}(x) = A_{ij} e^{\mu_{3j} x} + A_{2j} e^{-\mu_{3j} x} + A_{3j} \cos \mu_4 x + A_{4j} \sin \mu_4 x +$$

$$+ \sum_{m=0}^{\infty} \bar{\zeta}_{1m} Z_{jm} \cos \alpha_m x \quad (4.129)$$

$$\theta_{oj}(x) = B_{1j} \cosh \gamma_{2j} x + B_{2j} \sinh \gamma_{2j} x + B_{3j} \cos \gamma_{3j} x + B_{4j} \sin \gamma_{3j} x +$$

$$\sum_{m=0}^{\infty} \zeta_{2m} T_{jm} \cos \frac{m\pi x}{\ell} + zj \sum_{m=0}^{\infty} \zeta_{2m} Y_{jn} \cos \frac{m\pi x}{\ell} \quad (4.130)$$

where

$$\gamma_{4j} = \frac{\rho_j \omega^2}{E_j^x} , \quad \zeta_{4m} = \frac{1}{E_j^* A_j [\gamma_{4j}^2 - \alpha_m^2]} , \tag{4.131}$$

γ_{2j}, γ_{3j} are positive real quantities defined by

$$\gamma_{2j} = \left(\frac{(GJ)_j + \sqrt{(GJ)_j^2 + 4E_j^* C_{wj} \rho_j I_{oj} \omega^2}}{2E_j^* C_{wj}} \right)^{\frac{1}{2}} , \tag{4.132}$$

$$\gamma_{3j} = \left(\frac{-(GJ)_j + \sqrt{(GJ)_j^2 + 4E_j^* C_{wj} \rho_j I_{oj} \omega^2}}{2E_j^* C_{wj}} \right)^{\frac{1}{2}} , \tag{4.133}$$

and

$$u_{1j} = \left(\frac{-\rho}{2E_j^*} \omega^2 + \sqrt{\left(\frac{\rho}{E_j^*}\right)^2 \omega^4 + \frac{4\rho_j A_j}{I_{zj}} \omega^2} \right)^{\frac{1}{2}} , \tag{4.134}$$

$$u_{2j} = \left(\frac{\rho}{2E_j^*} \omega^2 + \sqrt{\left(\frac{\rho}{E_j^*}\right)^2 \omega^4 + \frac{4\rho_j A_j}{I_{yj}} \omega^2} \right)^{\frac{1}{2}} , \tag{4.135}$$

$$u_{3j} = \left\{\frac{\rho}{2E_j^*}\,\omega^2 + \sqrt{\left(\frac{\rho}{E_j^*}\right)^2 \omega^4 + \frac{4\rho_j A_j}{I_{yj}}\,\omega^2}\right\}^{\frac{1}{2}}, \tag{4.136}$$

$$u_{4j} = \left\{\frac{\rho}{2E_j^*}\,\omega^2 + \sqrt{\left(\frac{\rho}{E_j^*}\right)^2 \omega^4 + \frac{4\rho_j A_j}{I_{yj}}\,\omega^2}\right\}^{\frac{1}{2}}, \tag{4.137}$$

$$\bar{\zeta}_{om} = \frac{1}{E_j^* I_{zj}\alpha_m^4 - \rho_j I_{zj}\alpha_m^2 - \rho_j A_j\ \omega^2}\ , \tag{4.138}$$

$$\bar{\zeta}_{1m} = \frac{1}{E_j^* I_{yj}\alpha_m^4 - \rho_j I_{yj}\alpha_m^2 - \rho_j A_j\ \omega^2}\ , \tag{4.139}$$

and

$$\zeta_{2m} = \frac{1}{E_j^* C_{wj}\alpha_m^4 + (GJ)_j\alpha_m^2 - \rho_j I_{oj}\ \omega^2}\ . \tag{4.140}$$

If one neglects rotatory inertia of the stringers, then the solutions for $v_{oj}(x)$ and $w_{oj}(x)$ become:

$$v_{oj}(x) = D_{1j}\cosh\gamma_{oj}x + D_{2j}\sinh\gamma_{oj}x + D_{3j}\cos\gamma_{oj}x +$$

$$D_{4j}\sin \gamma_{oj}x + \sum_{m=0}^{\infty} \zeta_{om}Y_{jm}\cos \frac{m\pi x}{\ell} , \qquad (4.141)$$

$$w_{oj}(x) = A_{1j}\cosh \gamma_{1j}x + A_{2j}\sinh \gamma_{1j}x + A_{3j}\cos \gamma_{1j}x +$$

$$+ A_{4j}\sin \gamma_{1j}x + \sum_{m=0}^{\infty} \zeta_{1m}Z_{jm}\cos \frac{m\pi x}{\ell} , \qquad (4.142)$$

where

$$\gamma_{oj}^4 = \frac{\rho_j A_j \omega^2}{E_j^* I_{zj}} , \quad \zeta_{om} = \frac{1}{E_j^* I_{zj}(\alpha_m^4 - \gamma_{oj}^4)} , \qquad (4.143)$$

and

$$\gamma_{1j}^4 = \frac{\rho_j A_j \omega^2}{E_j^* I_{yj}} , \quad \zeta_{1m} = \frac{1}{E_j^* I_{yj}(\alpha_m^4 - \gamma_{1j}^4)} . \qquad (4.144)$$

For the case of a stringer with one axis of symmetry (i.e., $\sqcap$) such that ($I_{yz} = a_y = 0$) and $a_z \neq 0$, then the assumption is made that $a_z = 0$ and this case reduces back to case (1). The a_z term appears only in the m_{23} sub-matrix for the kinetic energy and for the stringers investigated ($\sqcap$) its effect was insignificant. If the stringers are equally-spaced, then the contribution of this term is zero anyway.

For the case of the nonsymmetrical section in which the shear center coincides with the centroid ($a_y = a_2 = 0$, $I_{yz} \neq 0$) then the bending vibrations in the two perpendicular directions are uncoupled with the torsional vibrations and the equations reduce to the problem of "double coupling." For this case, the characteristic equation will be a polynomial of eighth order and one must resort to an approximate solution. However, if one neglects rotatory inertia of the stringers, then for this case, Equations (4.120) and (4.121) become for the homogenous part:

$$(D^4-b_1)v + (b_2D^4)w = 0, \tag{4.145}$$

$$(b_4D^4)v + (D^4-b_3)w = 0, \tag{4.146}$$

where $D = \frac{d}{dx}$ and the constants b_1, b_2, b_3, and b_4 are given as follows:

$$b_1 = \frac{\rho_j A_j \omega^2}{E_j^* I_{zj}}, \quad b_2 = \frac{I_{yzj}}{I_{zj}}, \quad b_3 = \frac{\rho_j A_j \omega^2}{E_j^* I_{yj}}, \quad b_4 = \frac{I_{yzj}}{I_{yj}}. \tag{4.147}$$

Reduction of Equations (4.145) and (4.146) to "triangular form" yields the following two equations:

$$(D^4-b_1)v + b_2D^4w = 0, \tag{4.148}$$

$$[(1-b_2b_4)D^8 - (b_1+b_2)D^4 + b_1b_2]w = 0. \tag{4.149}$$

The characteristic equation for w can be written as:

$$r^8 - b_5r^4 + b_6 = 0$$

the constants b_5 and b_6 are given by:

$$b_5 = \frac{I_{yj}(E_j^{*2} I_{yzj} I_{zj} + \rho_j A_j \omega^2)}{(I_{yj} I_{zj} - I_{yzj}^2) E_j^*} , \tag{4.150}$$

and

$$b_6 = \frac{I_{yj}(I_{yzj}) \rho_j A_j \omega^2}{(I_{yj} I_{zj} - I_{yzj}^2) E_j^* I_{zj}} . \tag{4.151}$$

This equation can be solved by making the substitution $\lambda = r^4$ which yields the following quadratic equation

$$\lambda^2 - b_5 \lambda + b_6 = 0 \tag{4.152}$$

which has solutions

$$\lambda = \frac{b_5 \pm \sqrt{b_5^2 - 4b_6}}{2} . \tag{4.153}$$

Let us denote the discriminant by D_1. For a given set of stringer parameters and a given ω, one has the following three possibilities:

Case (i) if $D_1 < 0$, then the roots are complex conjugates

$$\lambda_{1,2} = \alpha_1 \pm i\alpha_3 , \tag{4.154}$$

Case (ii) if $D_1 > 0$, then the roots are real and unequal

$$\lambda_1 = \alpha_1, \quad \lambda_2 = \alpha_2, \tag{4.155}$$

Case (iii) if $D_1 = 0$, then the roots are real, equal,

$$\lambda_1 = \alpha_1, \quad \lambda_2 = \alpha_1. \tag{4.156}$$

The roots of Equation (4.152) for case (i) can be put in the form:

$$r_{1,2} = \pm(\gamma_5+i\gamma_6), \tag{4.157}$$

$$r_{3,4} = \pm(-\gamma_6+i\gamma_5), \tag{4.158}$$

$$r_{5,6} = \pm(\gamma_7+i\gamma_8), \tag{4.159}$$

$$r_{7,8} = \pm(-\gamma_8\pm i\gamma_7). \tag{4.160}$$

For case (ii), the roots can be written as:

$$r_{1,2} = \pm\gamma_5;\ r_{3,4} = \pm i\gamma_5;\ r_{5,6} = \pm\gamma_6;\ r_{7,8} = \ i\gamma_6 \tag{4.161}$$

and for case (iii), we have repeated roots of the form

$$r_{1,2} = \pm\gamma_5;\ r_{3,4} = \pm i\gamma_5;\ r_{5,6} = \pm\gamma_5;\ r_{7,8} = i\gamma_5. \tag{4.162}$$

For any of these cases, the general solution for w_{oj} may be written down. This solution is denoted by:

$$w_{oj}(x) = \sum_{i=1}^{8} A_{ij}w_i(x) \tag{4.163}$$

where $w_i(x)$ denote the linearly independent functions of the complementary solution. The solution for v may be recovered by standard techniques from Equation (4.148). The solution is denoted by:

$$v_{oj}(x) = \sum_{i=1}^{8} d_iA_{ij}v_i(x) \tag{4.164}$$

where $v_i(x)$ denote the functions occurring in the complementary solution and the d_i are known real constants which depend on the solution for $w_{oj}(x)$.

The complete solution is given by:

$$v_{oj}(x) = \sum_{i=1}^{8} d_iA_{ij}v_i(x) - \sum_{m=0}^{\infty} \overset{*}{\zeta}_{om}Y_{jm}\cos\alpha_m x$$

$$+ \sum_{m=0}^{\infty} \bar{\zeta}_{om}Z_{jm}\cos\alpha_m x, \tag{4.165}$$

$$w_{oj}(x) = \sum_{i=1}^{8} A_{ij}w_i(x) + \sum_{m=0}^{\infty} \bar{\zeta}_{1m}Y_{jm}\cos\alpha_m x$$

$$- \sum_{m=0}^{\infty} \overset{*}{\zeta}_{1m}Z_{jm}\cos\alpha_m x, \tag{4.166}$$

where

$$\zeta^*_{om} = (E^*_j I_{yzj} \alpha^4_m)/\det, \qquad (4.167)$$

and

$$\bar{\zeta}_{1m} = (E^*_j I_{yz} \alpha^4_m - \rho_j A_j \omega^2)/\det, \qquad (4.168)$$

$$\bar{\zeta}_{om} = (E^*_j I_{zj} \alpha^4_m - \rho_j A_j \omega^2)/\det, \qquad (4.169)$$

$$\zeta^*_{1m} = (E^*_j I_{yz} \alpha^4_m - \rho_j A_j \omega^2)/\det, \qquad (4.170)$$

and

$$\det = (E^*_j I_{zj} \alpha^4_m - \rho_j A_j \omega^2)(E^*_j I_{yj} \alpha^4_m - \rho_j A_j \omega^2) - E^{2*}_j \alpha^8_m I^2_{yzj}. \qquad (4.171)$$

Boundary Conditions for Stringers

For free vibration, the general solutions of the stringer equations will be required to satisfy homogenous boundary conditions such as any combination of clamped supported, simply supported or free supported ends. Applying a given set of boundary conditions enables one to express the arbitrary constants occurring in the general solutions in terms of the loading parameters as follows:

$$U_{ij} = \sum_{m=0}^{\infty} N_{im} \zeta_{4m} X_{jm}, \quad (i=1,2) \qquad (4.172)$$

$$B_{ij} = \sum_{m=0}^{\infty} G_{im}\zeta_{2m}T_{jm} + \sum_{m=0}^{\infty} G_{im}\zeta_{3m}Y_{jm}, \quad (i=1,2,3,4) \tag{4.173}$$

where the coefficients N_{im} and G_{im} have different forms for different boundary conditions. The arbitrary constants A_{ij} which occur in Equations (4.165) and (4.166) can be related to the external loading coefficients Z_{jm} and Y_{jm} in terms of a matrix equation as follows:

$$[B]_j \begin{Bmatrix} A_{ij} \\ A_{2j} \\ \vdots \\ A_{8j} \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_8 \end{Bmatrix}_j, \quad (j=1,2,\ldots,k) \tag{4.174}$$

where the coefficient matrix $[B]_j$ and the external loading matrices $\{\bar{Q}\}_j$ have different forms depending on the type of boundary conditions applied to the beam and the exact forms of the $V_i(x)$ and $W_i(x)$. Since the matrix [B] is nonsingular, one can express the unknown constants in terms of the applied loading by the following equation:

$$\begin{Bmatrix} A_{1j} \\ A_{2j} \\ A_{3j} \\ \vdots \\ A_{8j} \end{Bmatrix} = [B]_j^{-1} \begin{Bmatrix} Q_{1j} \\ Q_{2j} \\ \cdot \\ \cdot \\ \cdot \\ Q_{8j} \end{Bmatrix}, \quad j=1,2,\ldots,k \tag{4.175}$$

or

$$\begin{Bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{8j} \end{Bmatrix} = [\bar{u}] \begin{Bmatrix} Q_{1j} \\ Q_{2j} \\ \vdots \\ Q_{8j} \end{Bmatrix} , \ j=1,2,\ldots,k \tag{4.176}$$

where u^j_{ik} denotes the matrix element of $[B]^{-1}_j$ in the ith row and the kth column and $Q_{1j}, Q_{2j}, \ldots, Q_{8j}$ denote the applied loads. Equation (4.176) may be written as:

$$A_{ij} = \sum_{m=0}^{\infty} \sum_{k=1}^{8} u^j_{ik} \bar{K}_{im} Y_{jm} + \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} u^j_{ik} K_{im} Z_{jm} \tag{4.177}$$

where K_{im} and $\bar{K}_{im}$ will have different forms depending on the exact form of the boundary conditions employed. For the case when $I_{yz} = 0$ the arbitrary constants occurring in Equations (4.141) and (4.142) can be expressed in terms of the loading parameters by the following expressions:

$$A_{ij} = \sum_{m=0}^{\infty} K_{im} \zeta_{1m} Z_{jm}, \quad (i=1,2,3,4) \tag{4.178}$$

$$D_{ij} = \sum_{m=0}^{\infty} H_{im} \zeta_{om} Y_{jm}, \quad (i=1,2,3,4) \tag{4.179}$$

For this case it is possible to find closed form expressions for the quantities K_{im}, H_{im}, and G_{im} for any given homogenous boundary conditions.

For the particular case in which both ends of the stringer are fixed ($u = v = w = dw/dx = \theta = d\theta/dx = 0$ at $x = 0, \ell$), the H_{im}, K_{im}, and G_{im} have the following form:

(a) Clamped-clamped case (for H_{im})

$$H_{im} = F_{i1} + F_{i2}\cos m\pi, \text{ where}$$

$$F_{11} = \frac{1 - \cos \gamma_{oj}\ell \cosh \gamma_{oj}\ell - \sin \gamma_{oj}\ell \sinh \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.180)$$

$$F_{12} = \frac{\cosh \gamma_{oj}\ell - \cos \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.181)$$

$$F_{21} = \frac{\sinh \gamma_{oj}\ell \cos \gamma_{oj}\ell + \sin \gamma_{oj}\ell \cosh \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.182)$$

$$F_{22} = -\frac{\sin \gamma_{oj}\ell + \sinh \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.183)$$

$$F_{31} = \frac{1 + \sinh \gamma_{oj}\ell \sin \gamma_{oj}\ell - \cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.184)$$

$$F_{32} = -\frac{\alpha_m(\cos \gamma_{oj}\ell - \cosh \gamma_{oj}\ell)}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}, \quad (4.185)$$

$$F_{41} = - \frac{\sin \gamma_{oj}\ell \cosh \gamma_{oj}\ell + \sinh \gamma_{oj}\ell \cos \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)} , \qquad (4.186)$$

$$F_{42} = \frac{\sinh \gamma_{oj}\ell + \sin \gamma_{oj}\ell}{2(\cosh \gamma_{oj}\ell \cos \gamma_{oj}\ell - 1)}$$

(b) Clamped-clamped case (for K_{im})

$$K_{im} = E_{i1} + E_{i2} \cos m\pi, \text{ where}$$

$$E_{11} = \frac{1 - \cos \gamma_{1j}\ell \cosh \gamma_{1j}\ell - \sin \gamma_{1j}\ell \sinh \gamma_{1j}\ell}{2(\cosh \gamma_{1j}\ell \cos \gamma_{1j}\ell - 1)} , \qquad (4.188)$$

$$E_{12} = \frac{\cosh \gamma_{1j}\ell - \cos \gamma_{1j}\ell}{2(\cosh \gamma_{1j} \cos \gamma_{1j} - 1)} , \qquad (4.189)$$

$$E_{21} = \frac{\sinh \gamma_{1j}\ell \cos \gamma_{1j}\ell + \sin \gamma_{1j}\ell \cosh \gamma_{1j}\ell}{2(\cosh \gamma_{1j}\ell \cos \gamma_{1j}\ell - 1)} , \qquad (4.190)$$

$$E_{22} = - \frac{(\sin \gamma_{1j}\ell + \sinh \gamma_{1j}\ell)}{2(\cosh \gamma_{1j}\ell \cos \gamma_{1j}\ell - 1)} , \qquad (4.191)$$

$$E_{31} = \frac{1 + \sinh \gamma_{1j}\ell \sin \gamma_{1j}\ell - \cosh \gamma_{1j}\ell \cos \gamma_{1j}\ell}{2(\cosh \gamma_{1j}\ell \cos \gamma_{1j}\ell - 1)} , \qquad (4.192)$$

$$E_{32} = -\frac{(\cos\gamma_{ij}\ell - \cosh\gamma_{ij}\ell)\alpha_m}{2(\cosh\gamma_{ij}\ell\,\cos\gamma_{ij}\ell - 1)} \tag{4.193}$$

$$E_{41} = -\frac{\sin\gamma_{ij}\ell\,\cosh\gamma_{ij}\ell + \sinh\gamma_{ij}\ell\,\cos\gamma_{ij}\ell}{2(\cosh\gamma_{ij}\ell\,\cos\gamma_{ij}\ell - 1)}, \tag{4.194}$$

$$E_{42} = \frac{\sinh\gamma_{ij}\ell + \sin\gamma_{ij}\ell}{2(\cosh\gamma_{ij}\ell\,\cos\gamma_{ij}\ell - 1)}, \tag{4.195}$$

(c) Clamped-clamped case (for G_{im})

$$G_{im} = L_{i1} + L_{i2}\cos m\pi,$$

where

$$L_{11} = \frac{\gamma_{2j}\gamma_{3j} - \gamma_{2j}\gamma_{3j}\cosh\gamma_{2j}\cos\gamma_{3j} - \gamma_{3j}^2\sin\gamma_{3j}\ell\,\sinh\gamma_{2j}\ell}{2\gamma_{2j}\gamma_{3j}\cos\gamma_{3j}\ell\,\cosh\gamma_{2j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2-\gamma_{2j}^2)\sin\gamma_{3j}\,\sinh\gamma_{2j}\ell}, \tag{4.196}$$

$$L_{12} = \frac{\gamma_{2j}\gamma_{3j}(\cosh\gamma_{2j}\ell - \cos\gamma_{3j}\ell)}{2\gamma_{2j}\gamma_{3j}\cos\gamma_{3j}\ell\,\cosh\gamma_{2j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2-\gamma_{2j}^2)\sin\gamma_{3j}\ell\,\sinh\gamma_{2j}\ell}, \tag{4.197}$$

$$L_{21} = \frac{\gamma_{2j}\gamma_{3j}\sinh\gamma_{2j}\ell\,\cos\gamma_{3j}\ell + \gamma_{3j}^2\sin\gamma_{3j}\ell\,\cosh\gamma_{2j}\ell}{2\gamma_{2j}\gamma_{3j}\cos\gamma_{3j}\ell\,\cosh\gamma_{2j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2-\gamma_{2j}^2)\sin\gamma_{3j}\ell\,\sinh\gamma_{2j}\ell}, \tag{4.198}$$

(4.199)

$$L_{22} = -\frac{(\gamma_{3j}^2 \sin \gamma_{3j}\ell + \gamma_{2j}\gamma_{3j} \sinh \gamma_{2j}\ell)}{2\gamma_{2j}\gamma_{3j} \cos \gamma_{3j}\ell \cosh \gamma_{2j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2 - \gamma_{2j}^2)\sin \gamma_{3j}\ell \sinh \gamma_{2j}\ell},$$

(4.200)

$$L_{31} = \frac{\gamma_{2j}\gamma_{3j} + \gamma_{2j}^2 \sinh \gamma_{2j}\ell \sin \gamma_{3j}\ell - \gamma_{2j}\gamma_{3j} \cos \gamma_{3j}\ell \cosh \gamma_{2j}\ell}{2\gamma_{2j}\gamma_{3j} \cosh \gamma_{2j}\ell \cos \gamma_{3j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2 - \gamma_{2j}^2)\sin \gamma_{3j}\ell \sinh \gamma_{2j}\ell},$$

(4.201)

$$L_{32} = \frac{\gamma_{2j}\gamma_{3j}(\cos \gamma_{3j}\ell - \cosh \gamma_{2j}\ell)}{2\gamma_{2j}\gamma_{3j} \cosh \gamma_{2j}\ell \cos \gamma_{3j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2 - \gamma_{2j}^2)\sin \gamma_{3j}\ell \sinh \gamma_{2j}\ell},$$

(4.202)

$$L_{41} = -\frac{\gamma_{2j}^2 \cos \gamma_{3j}\ell \sinh \gamma_{2j}\ell + \gamma_{2j}\gamma_{3j} \sin \gamma_{3j}\ell \cosh \gamma_{2j}\ell}{2\gamma_{2j}\gamma_{3j} \cosh \gamma_{2j}\ell \cos \gamma_{3j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2 - \gamma_{2j}^2)\sin \gamma_{3j}\ell \sinh \gamma_{2j}\ell},$$

(4.203)

$$L_{42} = \frac{\gamma_{2j}^2 \sinh \gamma_{2j}\ell + \gamma_{2j}\gamma_{2j} \sin \gamma_{3j}\ell}{2\gamma_{2j}\gamma_{3j} \cosh \gamma_{2j}\ell \cos \gamma_{3j}\ell - 2\gamma_{2j}\gamma_{3j} + (\gamma_{3j}^2 - \gamma_{2j}^2)\sin \gamma_{3j}\ell \sinh \gamma_{2j}\ell},$$

where the subscript j may be omitted if all of the stringers are identical. The arbitrary constants U_{ij} and U_{2j} in Equation (4.127) are zero for the fixed-end case and therefore are not expressed in terms of the loading parameters X_{jm}.

The functions which occur in the general solutions of the stringer equations can be expanded as even functions over the interval $[0,\ell]$ in a uniformly convergent Fourier cosine series. The expansions for the functions $\cosh \gamma_{ij}x$, $\sinh \gamma_{ij}x$, $\cos \gamma_{ij}x$, and $\sin \gamma_{ij}x$ which occur in the general solution of case (1) are given by:

$$\cosh \gamma_{ij}x = \sum_{m=0}^{\infty} a_{im}^{j} \cos \alpha_m x, \tag{4.204}$$

$$\sinh \gamma_{ij}x = \sum_{m=0}^{\infty} b_{im}^{j} \cos \alpha_m x, \tag{4.205}$$

$$\cos \gamma_{ij}x = \sum_{m=0}^{\infty} C_{im}^{j} \cos \alpha_m x, \tag{4.206}$$

$$\sin \gamma_{ij}x = \sum_{m=0}^{\infty} c_{im}^{j} \cos \alpha_m x, \tag{4.207}$$

where

$$a_{io}^{j} = \frac{\sinh \gamma_{ij}\ell}{\gamma_{ij}\ell}, \tag{4.208}$$

$$b_{io}^{j} = \frac{(\cosh \gamma_{ij}\ell - 1)}{\gamma_{ij}\ell} \tag{4.209}$$

$$c_{io}^{j} = \frac{\sin \gamma_{ij}\ell}{\gamma_{ij}\ell}, \tag{4.210}$$

$$d^j_{io} = (1 - \cos \gamma_{ij} \ell)/\gamma_{ij} \ell, \tag{4.211}$$

$$a^j_{im} = \frac{2\ _{ij}(-1)^m \sinh(\gamma_{ij}\ )}{\ell(\gamma^2_{ij} + \alpha^2_m)}, \tag{4.212}$$

$$b^j_{im} = \frac{2\gamma_{ij}(\cosh(\gamma_{ij}\ell)(-1)^m - 1)}{\ell(\gamma^2_{ij} + \alpha^2_m)}, \tag{4.213}$$

$$c^j_{im} = \frac{2\gamma_{ij} \sin(\gamma_{ij}\ell)(-1)^m}{\ell(\gamma^2_{ij} + \alpha^2_m)}, \tag{4.214}$$

$$d^j_{im} = \frac{2\gamma_{ij}(1-\cosh \gamma_{ij}\ell)(-1)^m}{\ell(\gamma^2_{ij} + \alpha^2_m)}. \tag{4.215}$$

Therefore, for the case of no coupling the general solutions of the stringer equations are given by:

$$u_{oj}(x) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} N_{1\ell}\zeta_{4\ell}X_{j\ell}r^j_{4m} \sin(\alpha_m x) +$$

$$\sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} N_{2\ell}\zeta_{4\ell}X_{j\ell}r^j_{7m} \sin(\alpha_m x) +$$

$$\sum_{m=0}^{\infty} \zeta_{4m}X_{jm} \sin(\alpha_m x), \tag{4.216}$$

$$v_{oj}(x) = \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} H_{3\ell}\zeta_{o\ell}Y_j\ C^j_{om}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} H_{4\ell}\zeta_{o\ell}Y_{j\ell}d_{om}^{j}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty} \zeta_{om}Y_{jm}\cos(\alpha_m x), \tag{4.217}$$

$$w_{oj}(x) = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} K_{1\ell}\zeta_{1\ell}Z_{j\ell}a_{1m}^{j}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} K_{2\ell}\zeta_{1\ell}Z_{j\ell}b_{1m}^{j}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} K_{3\ell}\zeta_{1\ell}Z_{j\ell}C_{1m}^{j}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} K_{4\ell}\zeta_{1\ell}Z_{j\ell}d_{1m}^{j}\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty} \zeta_{1m}Z_{jm}\cos(\alpha_m x), \tag{4.218}$$

$$\theta_{oj}(x) = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} G_{1\ell}\zeta_{2\ell}a_{2m}^{j}(T_{j\ell}+Y_{j\ell})\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} G_{2\ell}\zeta_{2\ell}b_{2m}^{j}(T_{j\ell}+Y_{j\ell})\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} G_{3\ell}\zeta_{2\ell}C_{3m}^{j}(T_{j\ell}+Y_{j\ell})\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} G_{4\ell}\zeta_{2\ell}d_{3m}^{j}(T_{j\ell}+Y_{j\ell})\cos(\alpha_m x) +$$

$$\sum_{m=0}^{\infty} \zeta_{2m}(T_{jm}+Y_{jm})\cos(\alpha_m x). \tag{4.219}$$

Boundary Conditions for Shell

For free vibration, the general solutions of the shell equations will be required to satisfy homogenous boundary conditions of the form:

$$L_i\{u,v,w\} = 0, \quad (i=1,2,\ldots,8) \tag{4.220}$$

where the L_i are differential operators. Inserting Equations (4.116), (4.117), and (4.118) into the boundary conditions and using orthogonality conditions for sin $n\phi$ and cos $n\phi$ results in two sets of eight nonhomogenous equations for each integer n which relate the $\bar{C}_{sn}$ (symmetric part) and the $\bar{C}'_{sn}$ (antisymmetric part) to the applied loads $\{\bar{P}\}^s$ (symmetric) and $\{\bar{P}\}^a$ (antisymmetric), respectively. These equations may be written as two matrix equations as follows:

$$[\bar{D}] \begin{Bmatrix} \bar{C}_{1n} \\ \bar{C}_{2n} \\ \vdots \\ \bar{C}_{8n} \end{Bmatrix} = \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}^s , \quad (n=0,1,2,\ldots) \tag{4.221}$$

$$[\bar{D}] \begin{Bmatrix} \bar{C}'_{1n} \\ \bar{C}'_{2n} \\ \vdots \\ \bar{C}'_{8n} \end{Bmatrix} = \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}, \quad (n=0,1,2,\ldots) \qquad (4.222)$$

where the coefficient matrix $[\bar{D}]$ and the external loading matrices $\{\bar{P}\}^s$, $\{\bar{P}\}^a$ have different forms depending on the type of boundary conditions applied to the shell. For the particular case of clamped-clamped supported boundaries, the matrices $\{\bar{P}\}^s$ and $\{\bar{P}\}^a$ have the following forms:

$$
\{\bar{P}\}^{s} = \begin{Bmatrix} 0 \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_1 \cos n\phi_j Z_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_2 \sin n\phi_j\, T_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_4 f_3 \sin n\phi_j\, Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_8 f_4 \cos n\phi_j X_{jm} \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_1 \cos n\phi_j Z_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_2 \cos n\phi_j\, T_{jm} - \sum_{j=1}^{k}\sum_{n=0}^{\infty} g_2 f_3 \sin n\phi_j\, Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_7 f_4 \cos n\phi_j X_{jm} \\ 0 \\ 0 \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_1 (-1)^m \cos n\phi_j Z_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_2 \sin n\phi_j (-1)^m T_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_4 f_3 \sin n\phi_j (-1)^m Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_r g_8 f_4 \cos n\phi_j X_{jm} \\ 0 \end{Bmatrix} \quad (4.223)
$$

$$
\{\bar{P}\}^{a} = \begin{Bmatrix} 0 \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_1 \sin n\phi_j Z_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_2 \cos n\phi_j\, T_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_4 f_3 \cos n\phi_j\, Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_8 f_4 \sin n\phi_j\, X_{jm} \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_1 \sin n\phi_j\, Z_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_2 \sin n\phi_j\, T_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_2 f_3 \cos n\phi_j\, Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_7 f_4 \sin n\phi_j\, X_{jm} \\ 0 \\ 0 \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_1 (-1)^m \sin n\phi_j\, Z_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_3 f_2 \cos n\phi_j (-1)^m T_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_4 f_3 \cos n\phi_j (-1)^m Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} \delta_n g_8 f_4 \sin n\phi_j X_{jm} \\ -\sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_1 \sin n\phi_j (-1)^m Z_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_1 f_2 (-1)^m \sin n\phi_j\, T_{jm} + \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_2 f_3 (-1)^m \cos n\phi_j Y_{jm} - \sum_{j=1}^{k}\sum_{m=0}^{\infty} g_7 f_4 \cos n\phi_j (-1)^m X_{jm} \\ 0 \end{Bmatrix} \quad (4.224)
$$

Since the matrix $[\bar{D}]$ is nonsingular, one may express the unknown constants in terms of the applied loads as:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \vdots \\ \bar{C}_8 \end{Bmatrix} = [\bar{D}]^{-1} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}^s , \quad (n=0,1,2,\ldots) \qquad (4.225)$$

and

$$\begin{Bmatrix} \bar{C}'_1 \\ \bar{C}'_2 \\ \vdots \\ \bar{C}'_8 \end{Bmatrix} = [\bar{D}]^{-1} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}^a , \quad (n=0,1,2,\ldots) \qquad (4.226)$$

or

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \vdots \\ \bar{C}_8 \end{Bmatrix} = [\bar{\xi}] \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}^s , \quad (n=0,1,2,\ldots) \qquad (4.227)$$

and

$$\begin{Bmatrix} \bar{C}_1' \\ \bar{C}_2' \\ \vdots \\ \bar{C}_8' \end{Bmatrix} = [\xi] \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_8 \end{Bmatrix}, \quad (n=0,1,2,\ldots) \tag{4.228}$$

where ξ_{ij} denotes the matrix element if $[\bar{D}]^{-1}$.

Compatibility Relations

Each element of the composite structure has been described individually. To connect the thin-walled beams to the shell, we shall require that the following compatibility relations hold:

$$w_{oj}(x) = w(x,\phi_j), \quad (j=1,\ldots,K) \tag{4.229}$$

$$v_{oj}(x) = v(x,\phi_j) + z_j/a \frac{\partial w}{\partial \phi}(x,\phi_j), \quad (j=1,\ldots,K) \tag{4.230}$$

$$\theta_{oj}(x) = \frac{1}{a} \frac{\partial w}{\partial \phi}(x,\phi_j), \quad (j=1,\ldots,K) \tag{4.231}$$

$$u_{oj}(x) = u(x,\phi_j) + y_j \frac{\partial v}{\partial x}(x,\phi_j) + z_j \frac{\partial w}{\partial x}(x,\phi_j), \quad (j=1,\ldots,K) \tag{4.232}$$

where K is the total number of stringers attached to the shell. These compatibility relations relate the middle surface displacements of the shell at the line of attachment to the displacements of the shear center of the jth thin-walled rib.

Substitution of Equations (4.116), (4.117), (4.118) and Equations (4.216), (4.217), (4.218), (4.219) in conjunction with Equations (4.41) through (4.52) and Equations (4.227) and (4.228) into Equations (4.229), (4.230), (4.231), and (4.232) yields a doubly infinite system of simultaneous homogenous algebraic equations. These compatibility equations may be written in a single matrix equation as follows:

$$\left[\begin{array}{c|c|c|c} A_{QP} & B_{QP} & C_{QP} & K_{QP} \\ \hline E_{QP} & F_{QP} & G_{QP} & H_{QP} \\ \hline I_{QP} & J_{QP} & L_{QP} & M_{QP} \\ \hline N_{QP} & O_{QP} & U_{QP} & V_{QP} \end{array}\right] \begin{Bmatrix} \bar{Z}_p \\ \bar{T}_p \\ \bar{Y}_p \\ \bar{X}_p \end{Bmatrix} = \begin{Bmatrix} \bar{0} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{Bmatrix}, \tag{4.233}$$

where $\bar{Z}_p$, $\bar{T}_p$, $\bar{Y}_p$ and $\bar{X}_p$ are column vectors whose components are:

$$\bar{Z}_p = Z_j\ell, \tag{4.234}$$

$$\bar{T}_p = T_j\ell \tag{4.235}$$

$$\bar{Y}_p = Y_j\ell \tag{4.236}$$

$$\bar{X}_p = X_j\ell \tag{4.237}$$

and j,ℓ are related to P by

$$j = 1 + \left[(P-1)/\ell^*\right]_T, \quad (j=1,\ldots,K) \tag{4.238}$$

$$\ell = (P-1) - \ell^*\left[(P-1)/\ell^*\right]_T, \quad (\ell=1,2,\ldots,\ell^*) \tag{4.239}$$

where ℓ^* is the maximum value of ℓ and the subscript T represents the operation of truncation after the decimal point. Similarly, the submatrices $A_{QP}, B_{QP}, C_{QP}, \ldots, S_{QP}$ are given as follows:

$$A_{QP} = A_{imj\ell}, \tag{4.240}$$

$$B_{QP} = B_{imj\ell}, \tag{4.241}$$

$$C_{QP} = C_{imj\ell}, \tag{4.242}$$

$$K_{QP} = K_{imj\ell}, \tag{4.243}$$

$$\begin{matrix} \vdots & & \vdots \end{matrix}$$

$$S_{QP} = S_{imj\ell}, \tag{4.244}$$

and

$$i = 1 + [(Q-1)/\ell^*]_T, \tag{4.245}$$

$$m = (Q-1) - {}^*[(Q-1)/\ell^*]_T, \tag{4.246}$$

where P and Q are integers such that

$$1 \leq P \leq K\ell^{*}, \qquad (4.247)$$

$$1 \leq Q \leq K\ell^{*}, \qquad (4.248)$$

and K is the total number of stringers attached to the shell. For non-trivial solutions of $Z_{j\ell}$, $T_{j\ell}$, $Y_{j\ell}$, and $X_{j\ell}$, for J=1,...,K and ℓ=0,1,...,ℓ^{*} one requires that the determinant of the coefficient matrix of Equation (4.181) vanish. This yields the frequency equation, i.e.,

$$\begin{vmatrix} A_{QP} & B_{QP} & C_{QP} & K_{QP} \\ E_{QP} & F_{QP} & G_{QP} & H_{QP} \\ I_{QP} & J_{QP} & L_{QP} & M_{QP} \\ N_{QP} & O_{QP} & U_{QP} & V_{QP} \end{vmatrix} = 0. \qquad (4.249)$$

Equation (4.249) defines a determinant of order $[4K(\ell^{*}+1)] \times [4K(\ell^{*}+1)]$. For example, if there are four stringers attached to the shell and if ℓ=0,1,2,3 then the size of the determinant is (64) x (64). Once the natural frequencies are found, then calculations can be performed for the mode shapes.

Numerical Example

To illustrate the applicability of the general linear theory developed, a numerical example is presented. This example should also help to clarify some minor details which were omitted from the formal development of the solution. For the free vibration analysis, a clamped-clamped, longitudinally-stiffened, thin cylindrical shell is considered. The boundary conditions for the shell are given by:

$$\begin{aligned}
&\text{(i)}\ \ u(0,\phi) = 0, &&\text{(v)}\ \ u(\ell,\phi) = 0,\\
&\text{(ii)}\ \ v(0,\phi) = 0, &&\text{(vi)}\ \ v(\ell,\phi) = 0,\\
&\text{(iii)}\ \ w(0,\phi) = 0, &&\text{(vii)}\ \ w(\ell,\phi) = 0,\\
&\text{(iv)}\ \ \frac{\partial w}{\partial x}(0,\phi) = 0, &&\text{(viii)}\ \ \frac{\partial w}{\partial x}(\ell,\phi) = 0.
\end{aligned} \tag{4.250}$$

The boundary conditions for the thin-walled ribs are given by:

$$\text{(i)}\ \ u_{oj}(0) = u_{oj}(\ell) = 0,$$

$$\text{(ii)}\ \ v_{oj}(0) = v_{oj}(\ell) = 0,$$

$$\text{(iii)}\ \ \frac{dv_{oj}}{dx}(0) = \frac{dv_{oj}}{dx}(\ell) = 0,$$

$$\text{(iv)}\ \ w_{oj}(0) = w_{oj}(\ell) = 0,$$

$$\text{(v)} \quad \frac{dw_{oj}}{dx}(0) = \frac{dw_{oj}}{dx}(\ell) = 0,$$

$$\text{(vi)} \quad \theta_{oj}(0) = \theta_{oj}(\ell) = 0,$$

$$\text{(vii)} \quad \frac{d\theta_{oj}}{dx}(0) = \frac{d\theta_{oj}}{dx}(\ell) = 0, \tag{4.251}$$

where $j=1,\ldots,K$.

In Order to keep the size of the determinant in reasonable limits while investigating the free vibration characteristics of a stiffened shell, a cylinder with four equally spaced stringers of the type used in reference [4] (hat-shaped) was chosen. A computer program was written for the general case. The numerical results presented are for the first few circumferential and axial modes.

The coefficients $A_{QP}, B_{QP}, C_{QP}, \ldots, S_{QP}$ for the clamped-clamped case are given as follows:

$$A_{QP} = A_{imj\ell} = \{K_{1\ell}\xi_{1\ell}a_{im} + K_{2\ell}\xi_{1\ell}b_{im} + K_{3\ell}\xi_{1\ell}C_{1m} +$$

$$+ K_{4\ell}\xi_{1\ell}d_{1m} + \xi_{1m}\delta_{m\ell}\}\delta_{ij} + \sum_{n=0}^{N*}\sum_{s=1}^{8} \{P_{snj\ell} \cos n\phi_i +$$

$$P'_{snj\ell} \sin n\phi_i\}t_{5mn} - \sum_{n=0}^{N^*} h_4(\cos n\phi_j \cos n\phi_i +$$

$$\sin n\phi_j \sin n\phi_i)\delta_{\ell m}, \tag{4.252}$$

$$B_{QP} = B_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{Q_{snj\ell} \cos n\phi_i + Q'_{snj\ell} \sin n\phi_i\} \, t_{5mn} -$$

$$- \sum_{n=0}^{N^*} h_5 (\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i) \delta_{\ell m}, \tag{4.253}$$

$$C_{QP} = C_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{R_{snj\ell} \cos n\phi_i + R'_{snj\ell} \sin n\phi_i\} t_{5mn} -$$

$$\sum_{n=0}^{N^*} h_6 (\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i) \delta_{\ell m}, \tag{4.254}$$

$$K_{QP} = K_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{S_{snj\ell} \cos n\phi_i + S'_{snj\ell} \sin n\phi_i\} t_{5mn} -$$

$$\sum_{n=0}^{N^*} h_{10} (\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i) \delta_{\ell m}, \tag{4.255}$$

$$E_{QP} = E_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{P_{snj\ell} \sin n\phi_i + P'_{snj\ell} \cos n\phi_i\} a_{5mn}$$

$$- \sum_{n=0}^{N^*} h_1 (\cos n\phi_j \sin n\phi_i + \sin n\phi_j \cos n\phi_i) \delta_{\ell m},$$

$$+ \sum_{n=0}^{N^*} h_4 Z_i (n/2) (\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \delta_{\ell m} \tag{4.256}$$

$$F_{QP} = F_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{Q_{snj\ell} \sin n\phi_i + Q'_{snj\ell} \cos n\phi_i\} (a_{5mn})$$

$$- \sum_{n=0}^{N^*} h_2 (\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i) \delta_{\ell m} +$$

$$+ \sum_{n=0}^{N^*} h_5 Z_i (n/a)(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)\delta_{\ell m}, \quad (4.257)$$

$$G_{QP} = G_{imj\ell} = \{H_{1\ell}\zeta_{o\ell}a_{om} + H_{2\ell}\zeta_{o\ell}b_{om} + H_{3\ell}\zeta_{o\ell}C_{om} + H_{4\ell}\zeta_{o\ell}d_{om} +$$

$$\zeta_{om}\delta_{m\ell}\}\delta_{ij} + \sum_{n=0}^{N^*}\sum_{s=1}^{8} \{R_{snj\ell} \sin n\phi_i + R'_{snj\ell} \cos n\phi_i\}(a_{smn}) -$$

$$- \sum_{n=0}^{N^*} h_3(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i)\delta_{\ell m}$$

$$+ \sum_{n=0}^{N^*} h_2 Z_8 (n/a)(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i)\delta_{\ell m}, \quad (4.258)$$

$$H_{QP} = H_{imj\ell} = \sum_{n=0}^{N^*}\sum_{s=1}^{8} \{S_{snj\ell} \sin n\phi_i + S'_{snj\ell} \cos n\phi_i \; (a_{smn})$$

$$- \sum_{n=0}^{N^*} h_{12}(\cos n\phi_i \sin n\phi_j - \sin n\phi_i \cos n\phi_j)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_{10}(\sin n\phi_i \cos n\phi_j + \cos n\phi_i \sin n\phi_j)\delta_{\ell m}(n/a)z_i, \quad (4.259)$$

$$I_{QP} = I_{imj\ell} = \sum_{n=0}^{N^*}\sum_{s=1}^{8} \{P_{snj\ell} \sin n\phi_i + P'_{snj\ell} \cos n\phi_i\}(n/a)t_{5mn} -$$

$$\sum_{n=1}^{N^*} h_4\left(\frac{z_i}{a}\right)(n)(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)\delta_{\ell m}, \quad (4.260)$$

$$J_{QP} = J_{imj\ell} = \{G_{1\ell}\zeta_2 \, a_{2m} + G_{2\ell}\zeta_{2\ell}b_{2m} + G_{3\ell}\zeta_{2\ell}C_{3m} + G_{4\ell}\zeta_{2\ell}d_{3m} +$$

$$\zeta_{2m}\delta_{m\ell}\}\delta_{ij} + \sum_{n=1}^{N^*}\sum_{s=1}^{8} \{Q_{snj\ell} \sin n\phi_i + Q'_{snj\ell} \cos n\phi_i\}(n/a)t_{5mn} -$$

$$\sum_{n=1}^{N^*} h_5\left(\frac{z_i}{a}\right)(n)\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)\delta_{\ell m}, \qquad (4.261)$$

$$L_{QP} = L_{imj\ell} = \{G_{1\ell}\zeta_{2\ell}a_{2m} + G_{2\ell}\zeta_{2\ell}b_{2m} + G_{3\ell}\zeta_{2\ell}C_{3m} + G_{4\ell}\zeta_{2\ell}d_{3m} +$$

$$\zeta_{2m}\delta_{m\ell}\}\delta_{ij} + \sum_{n=1}^{N^*}\sum_{s=1}^{8}\{R_{snj\ell} \sin n\phi_i + R'_{snj\ell} \cos n\phi_i\}(n/a)t_{5mn} -$$

$$\sum_{n=1}^{N^*} h_6\left(\frac{z_i}{a}\right)(n)(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i)\delta_{\ell m}, \qquad (4.262)$$

$$M_{QP} = M_{imj\ell} - \sum_{n=1}^{N^*}\sum_{s=1}^{8} \{S_{snj\ell} \sin n\phi_i + S'_{snj\ell} \cos n\phi_i\}\left(\frac{z_i}{a}\right)(n)t_{5mn}$$

$$+ \sum_{n=0}^{N^*} h_{12}(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)\left(\frac{z_i}{a}\right)(n)$$

$$+ \sum_{n=0}^{N^*} h_{11}(n)(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)\left(\frac{z_i}{a}\right), \qquad (4.263)$$

$$N_{QP} = N_{imj\ell} = \sum_{n=0}^{N^*}\sum_{s=1}^{8} \{P_{snj\ell} \cos n\phi_i\{b_{smn} - (z_i\alpha_m + y_i\alpha_m)t_{smn}\} +$$

$$P'_{snj\ell} \sin n\phi_i\{b_{smn} - (z_i\alpha_m - y_i\alpha_m)t_{smn}\}\ \}$$

$$+ \sum_{n=1}^{N^*} h_4 z_i\alpha_m(\sin n\phi_j \cos n\phi_i + \cos n\phi_j \sin n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_7 z_i\alpha_m(\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)\delta_{\ell m}$$

$$+ \sum_{n=1}^{\infty} h_1 y_i\alpha_m(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)\delta_{\ell m}, \qquad (4.264)$$

$$O_{QP} = O_{imj\ell} = \sum_{n=0}^{N^*} \sum_{s=1}^{8} \{Q_{snj\ell} \cos n\phi_i \{b_{smn} - (z_i\alpha_m + y_i\alpha_m)t_{smn}\} +$$

$$Q'_{snj\ell} \sin n\phi_i \{b_{smn} - (z_i\alpha_m - y_i\alpha_m)t_{smn}\} \}$$

$$+ \sum_{n=0}^{N^*} h_5 z_i\zeta_m(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_8(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_2 y_i\alpha_m(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i)\delta_{\ell m}, \qquad (4.265)$$

$$U_{QP} = U_{imj\ell} = \sum_{n=1}^{N^*} \sum_{s=1}^{8} \{R_{snj\ell} \cos n\phi_i \{b_{smn} - (z_i\alpha_m + y_i\alpha_m)t_{smn}\} +$$

$$R'_{snj\ell} \sin n\phi_i \{b_{smn} - (z_i\alpha_m - y_i\alpha_m)t_{smn}\} \}$$

$$+ \sum_{n=0}^{N^*} h_6 z_i\alpha_m(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_9(\cos n\phi_j \sin n\phi_i + \sin n\phi_j \cos n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=0}^{N^*} h_3 y_i\alpha_m(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i)\delta_{\ell m}, \qquad (4.266)$$

$$V_{QP} = V_{imj\ell} = \{N_{1\ell}\zeta_{4\ell}r_{4m} + N_{2\ell}\zeta_{4\ell}r_{7m} + \zeta_{4m}\delta_{m\ell}\}\delta_{ij} +$$

$$\sum_{n=0}^{N^*} \sum_{s=1}^{8} \{S_{snj\ell} \cos n\phi_i \{b_{smn} - (z_i\alpha_m + y_i\alpha_m)t_{smn}\} +$$

$$S'_{snj\ell} \sin n\phi_i \{b_{smn} - (z_i\alpha_m - y_i\alpha_m)t_{smn}\} \}$$

$$- \sum_{n=0}^{N^*} h_{11}(\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)\delta_{\ell m} -$$

$$+ \sum_{n=1}^{N^*} h_{10} z_i\alpha_m(\sin n\phi_j \cos n\phi_i + \cos n\phi_j \sin n\phi_i)\delta_{\ell m}$$

$$- \sum_{n=1}^{N^*} h_{12} y_i\alpha_m(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)\delta_{\ell m}, \qquad (4.267)$$

where the following notation has been introduced:

$$g_3 f_1 = h_1, \qquad (4.268)$$

$$g_3 f_2 = h_2, \qquad (4.269)$$

$$g_4 f_3 = h_3, \qquad (4.270)$$

$$g_1 f_1 = h_4, \qquad (4.271)$$

$$g_1 f_2 = h_5, \qquad (4.272)$$

$$g_2 f_3 = h_6, \qquad (4.273)$$

$$g_5 f_1 = h_7, \qquad (4.274)$$

$$g_5 f_2 = h_8, \qquad (4.275)$$

$$g_6 f_3 = h_9, \tag{4.276}$$

$$g_7 f_4 = h_{10}, \tag{4.277}$$

$$g_9 f_4 = h_{11}, \tag{4.278}$$

$$g_8 f_4 = h_{12}, \tag{4.279}$$

$$a_{1mn} = t_{1mn}\left(\bar{\beta}_{1n} - \frac{Z_j n}{a}\right) - t_{2mn}\bar{\beta}_{2n}, \tag{4.280}$$

$$a_{2mn} = t_{1mn}\bar{\beta}_{2n} + t_{2mn}\left(\bar{\beta}_{1n} - \frac{Z_j n}{a}\right), \tag{4.281}$$

$$a_{3mn} = t_{3mn}\left(\bar{\beta}_{3n} - \frac{Z_j n}{a}\right) - t_{4mn}\bar{\beta}_{4n}, \tag{4.282}$$

$$a_{4mn} = t_{3mn}\bar{\beta}_{4mn} + t_{4mn}\left(\bar{\beta}_{3n} - \frac{Z_j u}{2}\right), \tag{4.283}$$

$$a_{5mn} = t_{5mn}\left(\bar{\beta}_{5n} - \frac{Z_j u}{a}\right) - t_{6mn}\bar{\beta}_{6n}, \tag{4.284}$$

$$a_{6mn} = t_{5mn}\bar{\beta}_{6n} + \bar{t}_{6mn}\left(\bar{\beta}_{5n} - \frac{Z_j n}{a}\right), \tag{4.285}$$

$$a_{7mn} = t_{7mn}\left(\bar{\beta}_{7n} - \frac{Z_j n}{a}\right) - t_{8mn}\bar{\beta}_{8n}, \tag{4.286}$$

$$a_{8mn} = t_{7mn}\bar{\beta}_{8n} + t_{8mn}\left(\bar{\beta}_{7n} - \frac{Z_j n}{a}\right), \tag{4.287}$$

$$b_{1mn} = \bar{\alpha}_{1n} r_{1mn} - \bar{\alpha}_{2n} r_{2mn}, \tag{4.288}$$

$$b_{2mn} = \bar{\alpha}_{2n} r_{1mn} + \bar{\alpha}_{1n} r_{2mn}, \tag{4.289}$$

$$b_{3mn} = \bar{\alpha}_{3n} r_{3mn} - \bar{\alpha}_{4n} r_{4mn}, \tag{4.290}$$

$$b_{4mn} = \bar{\alpha}_{4n} r_{3mn} + \bar{\alpha}_{3n} r_{4mn}, \tag{4.291}$$

$$b_{5mn} = \bar{\alpha}_{5n} \bar{r}_{5mn} - \bar{\alpha}_{6n} r_{6mn}, \tag{4.292}$$

$$b_{6mn} = \bar{\alpha}_{6n} \bar{r}_{5mn} + \bar{\alpha}_{5n} r_{6mn}, \tag{4.293}$$

$$b_{7mn} = \bar{\alpha}_{7n} r_{7mn} + \bar{\alpha}_{8n} r_{8mn}, \tag{4.294}$$

$$b_{8mn} = \bar{\alpha}_{8n} r_{7mn} + \bar{\alpha}_{7n} r_{8mn}, \tag{4.295}$$

$$\begin{aligned} P_{snj\ell} &= \xi_{32} \delta_n h_1 \cos n\phi_j + \xi_{53} h_4 \cos n\phi_j + \xi_{56} \delta_n h_1 (-1)^\ell \cos n\phi_j \\ &\quad + \xi_{57} h_4 (-1)^\ell \cos n\phi_j, \end{aligned} \tag{4.296}$$

$$\begin{aligned} P'_{snj\ell} &= \xi_{52} \delta_n h_1 \sin n\phi_j + \xi_{53} h_4 \sin n\phi_j + \xi_{56} \delta_n h_1 (-1)^\ell \sin n\phi_j \\ &\quad + \xi_{57} h_4 (-1)^\ell \sin n\phi_j, \end{aligned} \tag{4.297}$$

$$\begin{aligned} Q_{snj\ell} &= \xi_{52} h_2 \delta_n \sin n\phi_j + \gamma_{53} h_5 \cos n\phi_j + \xi_{56} \delta_n h_2 \sin n\phi_j (-1)^\ell \\ &\quad + \xi_{57} h_5 \cos n\phi_j (-1)^\ell, \end{aligned} \tag{4.298}$$

$$Q'_{snj\ell} = -\xi_{52}h_2\delta_n \cos n\phi_j + \xi_{53}h_5 \sin n\phi_j$$

$$- \xi_{56}\delta_n h_2 \cos n\phi_j(-1)^\ell + \xi_{57}h_5 \sin n\phi_j(-1)^\ell, \quad (4.299)$$

$$R_{snj\ell} = \xi_{52}h_3\delta_n \sin n\phi_j + \xi_{53}h_6 \sin n\phi_j +$$

$$+ \xi_{56}\delta_n h_3(-1)^\ell \sin n\phi_j + \xi_{57}h_6 \sin n\phi_j(-1)^\ell, \quad (4.300)$$

$$R'_{snj\ell} = -\xi_{52}h_3\delta_n \cos n\phi_j - \xi_{53}h_6 \cos n\phi_j -$$

$$- \xi_{56}\delta_n h_3(-1)^\ell \cos n\phi_j - \xi_{57}h_6 \cos n\phi_j(-1)^\ell, \quad (4.301)$$

$$S_{snj\ell} = \xi_{52}h_{12}\delta_n \cos n\phi_j + \xi_{53}h_{10} \cos n\phi_j$$

$$+ \xi_{56}\delta_n h_{12} \cos n\phi_j(-1)^\ell + \xi_{57}h_{10} \cos n\phi_j(-1)^\ell, \quad (4.302)$$

$$S_{snj\ell} = \xi_{52}h_{12}\delta_n \sin n\phi_j + \xi_{53}h_{10} \sin n\phi_j$$

$$+ \xi_{56}\delta_n h_{12} \sin n\phi_j(-1)^\ell + \xi_{57}h_{10} \sin n\phi_j(-1)^\ell, \quad (4.303)$$

where

$$\delta_n = 0 \text{ if } n = 0 \quad (4.304)$$

$$= 1 \text{ if } n \neq 0.$$

Numerical Results and Discussion

Computer programs were written to solve the eigenvalue problem (i.e., (4.249)) in the ALGOL for use with the Burrough's B5500 digital computer. Two numerical examples are considered. The first is for a simply supported cylindrical shell with four and eight internal stringers and the second is for a clamped-clamped shell with four internal stringers.

Free Vibration

Table 22 and Table 23 present a comparison of the theoretical frequencies as predicted by the general linear theory with the results of the energy method for four and eight equally spaced stringers on a simply supported cylindrical shell for both symmetric and antisymmetric modes. The theoretical calculations employed a 30-term series for the circumferential modes. Comparison of 25- and 30-term expansions showed showed excellent agreement for frequencies calculated ($n \leq 10$). The comparison of the frequencies for the simply supported stiffened shell shows good agreement between both methods of solution. It should be noted between both methods of solution. It should be noted that for the general theory there are some slight differences in frequencies between odd symmetric and antisymmetric circumferential modes (i.e., $m=1$, $n=7$). Theoretically, these frequencies should be the same. The trouble stems from the approximation made of the twisting moment acting on the stringer. It was found that slightly better results could be obtained by considering a couple acting on the shell.

Figure 13 shows a comparison of some selected modes shapes as predicted by the general theory with the results of the Ritz method for four internal hat-shaped stringers. In general, very good agreement is noted between the two different methods. Better convergence of the modes was noted for the general linear theory as compared with the Ritz analysis (i.e., m=1, n=1). The results are in more disagreement for comparison between the antisymmetric circumferential modes. Again, this is attributed to the mathematical representation of the twisting moment. Slightly better results may be obtained by expressing the twisting moment in terms of a singularity function.

Figure 14 shows a comparison of some selected circumferential mode shapes as predicted by the general linear theory with the results of the Ritz method for eight internal hat-shaped stringers. Again the results of the general theory show much better convergence properties than the Ritz method for the same number of terms in the series. The eigenvalues as predicted from the general theory were obtained to six decimal place accuracy as with only two decimal place accuracy from the Ritz method for the modes n=1,3. The mode shapes from the general theory are relatively smooth functions indicating that not enough terms were taken in the Ritz method to achieve a good approximation of the mode.

Table 24 shows a comparison of theoretical frequencies of the general theory for a clamped-clamped shell with four equally spaced flexible stringers versus those of the unstiffened shell using Donnell's shell theory for the lower axial and circumferential modes. Frequencies are presented for both symmetric and antisymmetric modes. For the example considered, the stringers were very weak in torsion. This is reflected in the numerical results. The frequencies for the even antisymmetric circumferential modes are very nearly equal to those of the unstiffened cylindrical shell. These frequencies are slightly less than those for the unstiffened cylinder since the stringers contribute inertia to the structure but the contribution to the potential energy is insignificant since the stringers are so weak in torsion.

The frequencies for the even symmetric modes for the case when n=4 are greater than those of the unstiffened shell since all the stringers are in bending and contribute more potential energy to the structure than kinetic energy. Theoretical mode shapes for the axial modes (m=1,2) and some lower circumferential modes are presented in Figure 15. Theoretical calculations employed six terms for the axial mode and 20 terms for the circumferential mode. It is necessary to take more terms to represent the circumferential mode shape since the presence of the stringers on the shell is to couple the unstiffened shell modes together as is seen from the Ritz analysis for the simply supported shell. Fewer terms are needed to represent the lower axial modes since these are relatively smooth curves. It should be noted that the integer n in Table 23 stands for the number of full waves around the circumference.

Table 22. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Four Equally-Spaced Stringers

m=1 n	a	b	c	d	m=2 n	a	b	c	d
	f in cps					f in cps			
1	745	721	745	721	1	1829	1827	1829	1827
2	317	315	302	301	2	984	985	959	962
3	157	157	157	157	3	546	548	546	549
4	105	104	99	99	4	356	357	337	340
5	91	91	91	91	5	243	243	243	243
6	106	107	109	109	6	204	207	189	191
7	137	137	137	135	7	187	188	187	188
8	173	174	181	179	8	207	216	204	211
9	222	222	222	222	9	238	240	238	240
10	265	264	279	277	10	281	284	286	287
m=3 n					m=4 n				
1	2648	2646	2648	2646	1	3136	3134	3136	3134
2	1637	1643	1647	1654	2	2188	2189	2188	2189
3	1039	1045	1039	1045	3	1519	1533	1519	1534
4	721	725	673	682	4	1072	1086	816	821
5	491	505	491	505	5	781	783	781	783
6	396	400	355	368	6	639	642	592	603
7	313	329	313	329	7	488	492	488	491
8	302	306	273	277	8	424	436	391	403
9	288	287	288	286	9	384	391	384	391
10	321	322	315	318	10	402	412	371	379

a = Ritz analysis (symmetric modes).

b = General theory (symmetric modes).

c = Ritz analysis (antisymmetric modes).

d = General theory (antisymmetric modes).

Table 23. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Eight Equally-Spaced Stringers

m=1	f in cps				m=2	f in cps			
n	a	b	c	d	n	a	b	c	d
1	737	734	737	734	1	1918	1923	1918	1923
2	313	311	313	312	2	970	973	970	971
3	160	157	160	157	3	556	559	556	559
4	109	108	99	99	4	359	363	331	334
5	91	91	92	91	5	250	249	250	250
6	105	104	105	104	6	203	205	203	207
7	133	132	133	133	7	191	198	200	198
m=3					m=4				
n					n				
1	2587	2583	2587	2583	1	3136	3149	3136	3151
2	1622	1655	1622	1655	2	2188	2195	2168	2195
3	1047	1068	1047	1067	3	1519	1527	1519	1527
4	724	735	668	681	4	1072	1105	1138	1163
5	504	514	504	516	5	781	786	781	787
6	392	409	392	412	6	639	651	639	653
7	337	346	337	349	7	488	490	488	494

a = Ritz analysis (symmetric modes).

b = General theory (symmetric modes).

c = Ritz analysis (antisymmetric modes).

d = General theory (antisymmetric modes).

Table 24. Theoretical Frequencies of General Theory for a Clamped-Clamped Shell with Four Equally-Spaced Stringers

m=1	f(cps)			m=2	f(cps)		
n	a	b	c	n	a	b	c
1	2281	2283	2329	1	4551	4554	4772
2	1454	1427	1429	2	2737	2678	2682
3	2329	2330	2336	3	2773	2773	2771
4	4145	4137	4142	4	4361	4333	4335

a = General theory (stiffened) symmetric modes.

b = General theory (stiffened) antisymmetric modes.

c = Unstiffened shell (Donnell's theory) [12].

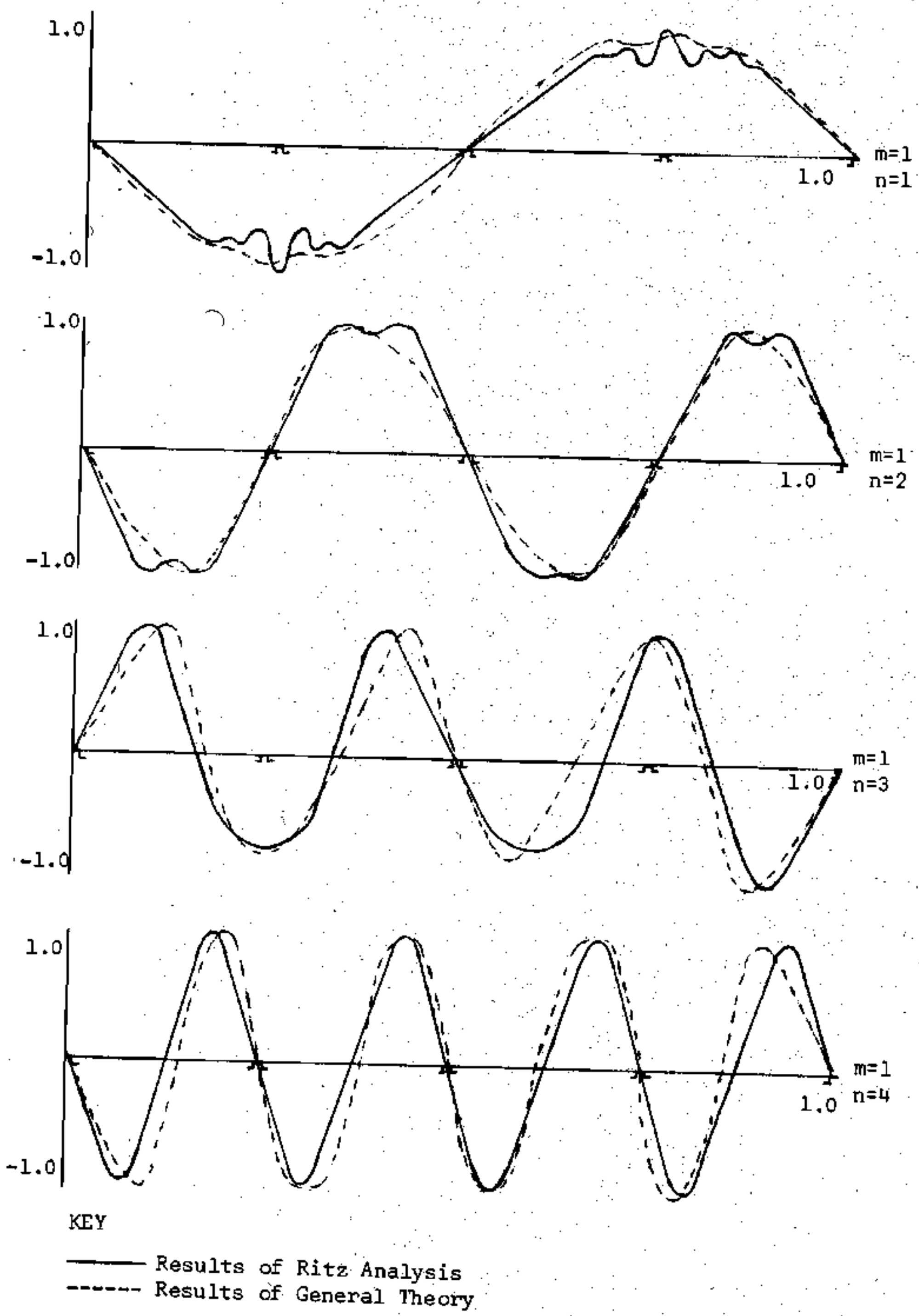


Figure 13. Theoretical Mode Shapes for Four Stringers
(Simply Supported Case)

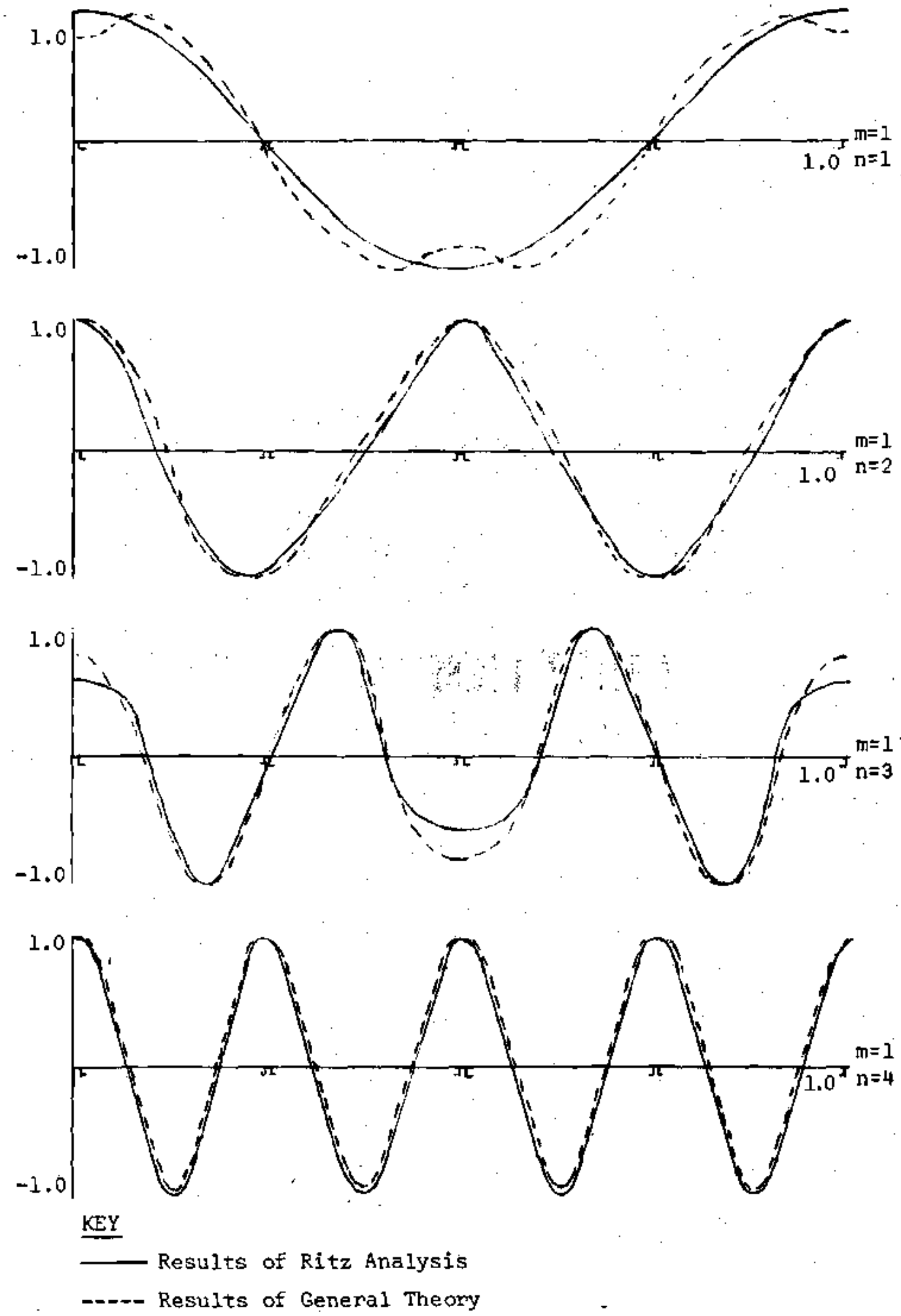
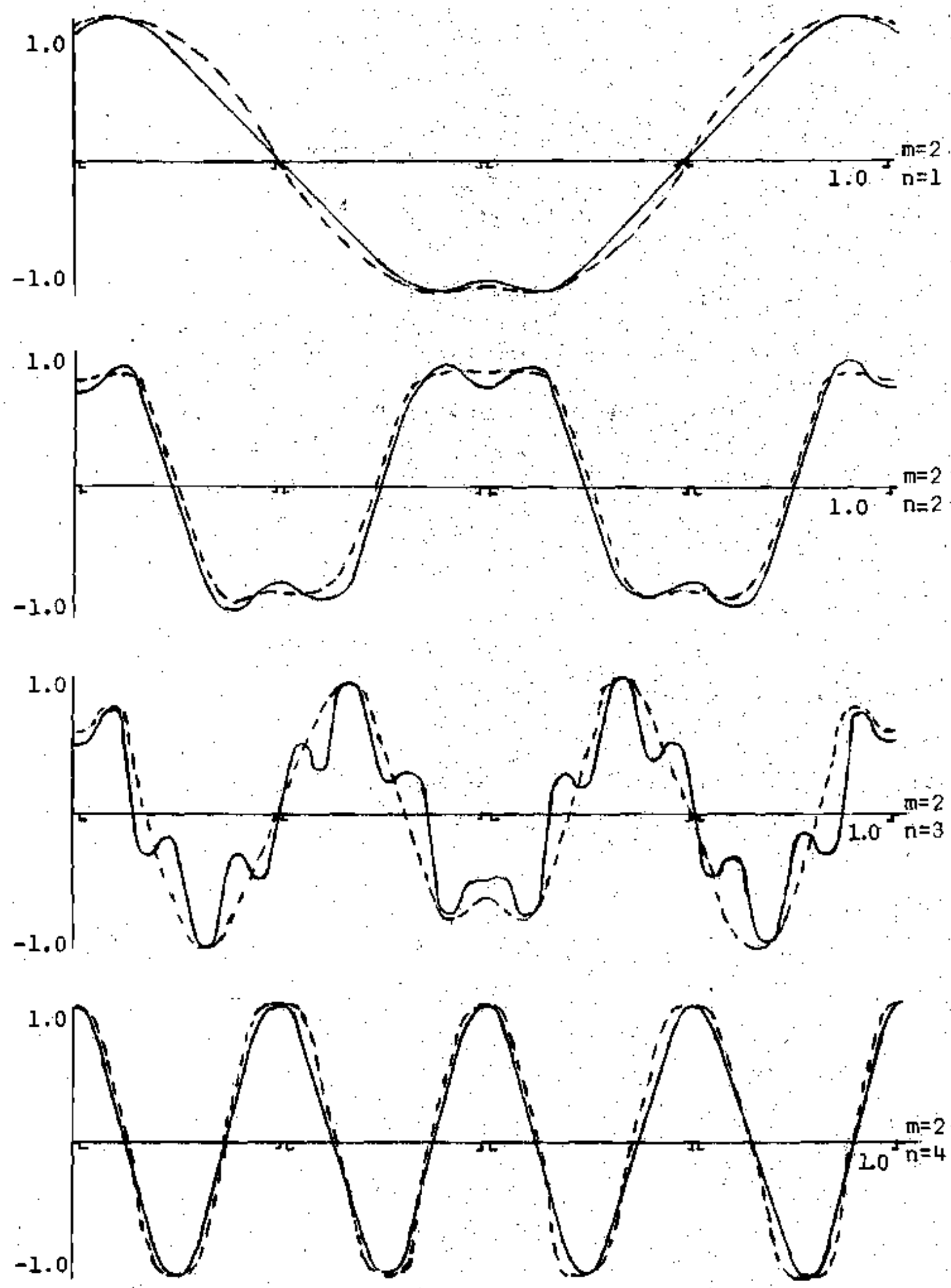


Figure 13. Theoretical Mode Shapes for Four Stringers (Symmetric Modes) (Continued)



KEY

——— Results of Ritz Analysis
------ Results of General Theory

Figure 13. Theoretical Mode Shapes for Four Stringers (Symmetric Modes) (Continued)

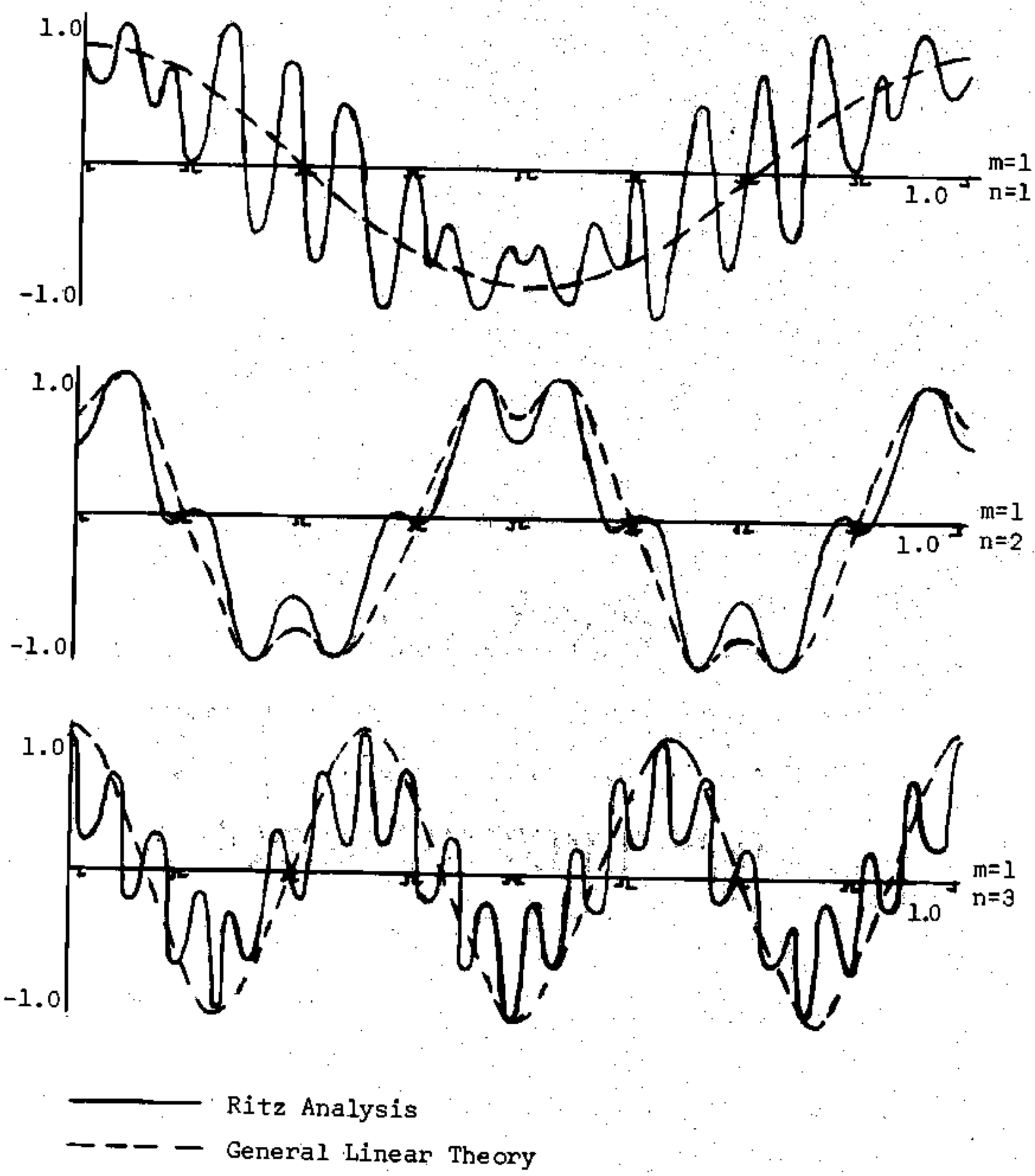


Figure 14. Theoretical Circumferential Mode Shapes for Eight Stringers (Simply Supported Case)

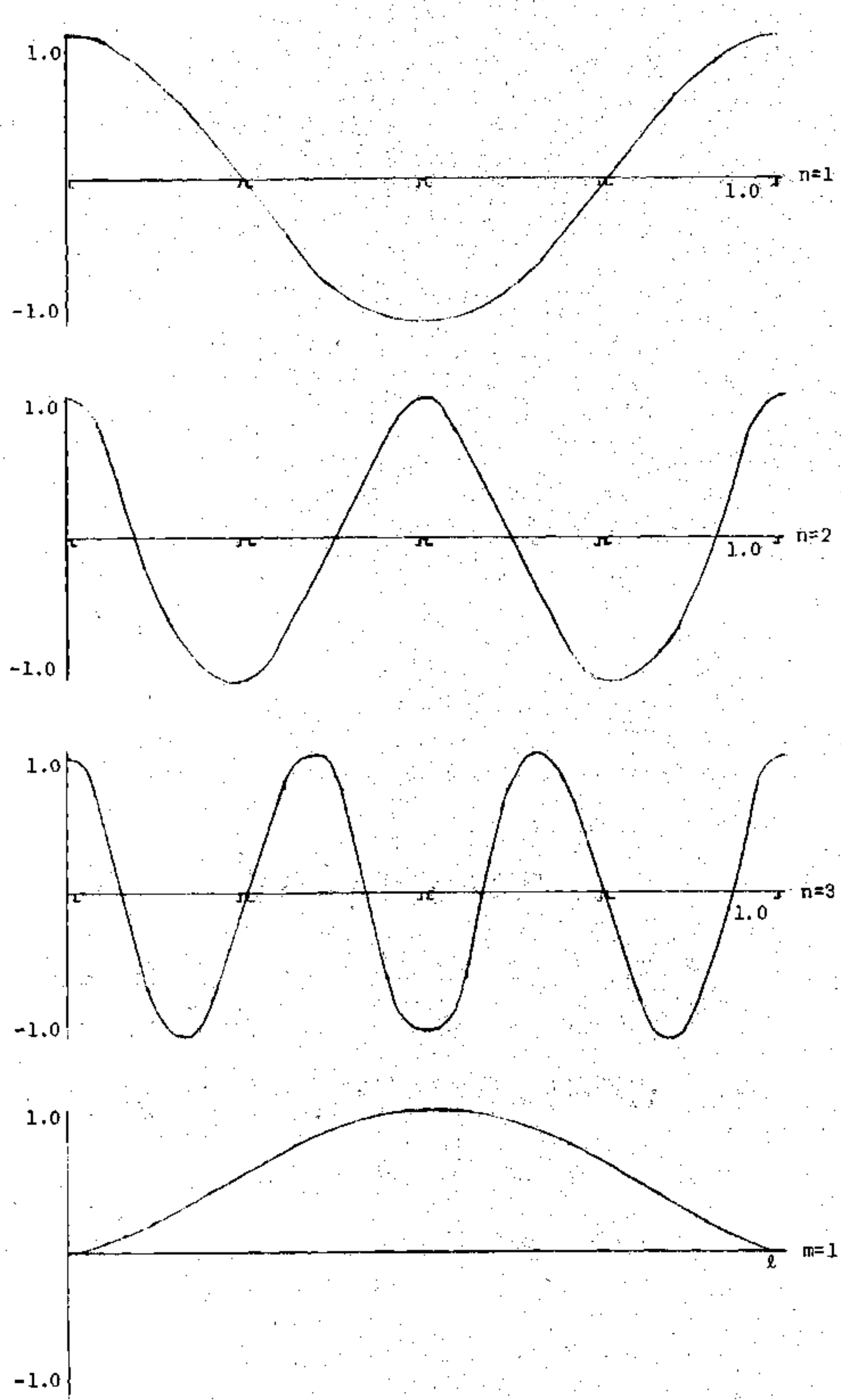


Figure 15. Theoretical Axial and Circumferential Modes for a Clamped-Clamped Cylinder with Four Stringers

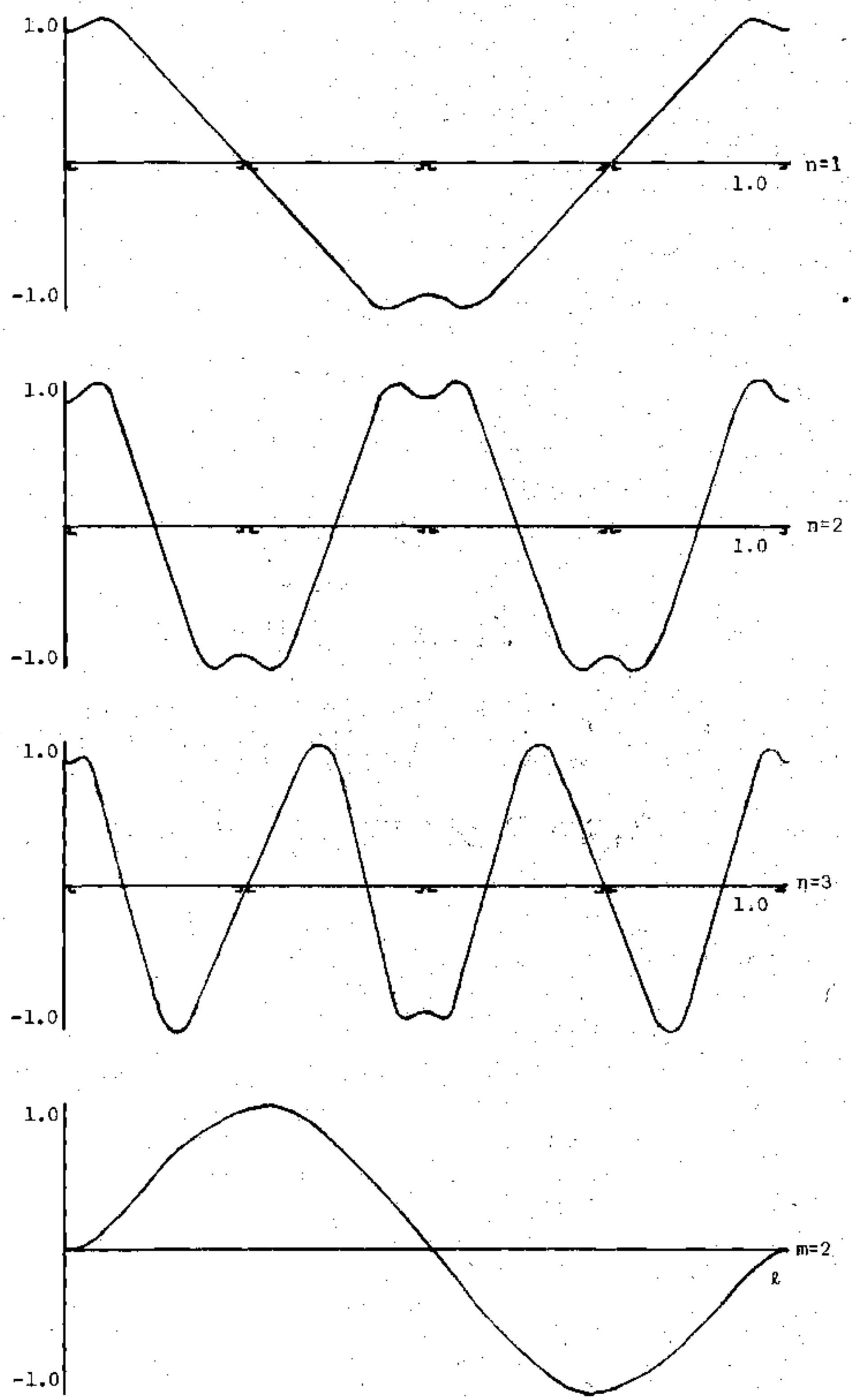


Figure 15. Theoretical Axial and Circumferential Mode Shapes for Clamped-Clamped Cylinder with Four Stringers (Continued)

CHAPTER V

SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS

Summary of Conclusions

A theoretical analysis has been performed to predict the free vibration characteristics of a thin cylindrical shell stiffened by means of longitudinal ribs for any set of admissible boundary conditions. A mathematical model was developed using Donnell's linear thin shell theory and V. Z. Vlasov's equations for the stringers. The stringers are treated as discrete structural elements and are not assumed to be closely or equally spaced. Thus, there are no approximations due to "smearing" the stringer effects over the surface of the shell.

It was shown that for a shell with only a few stringers, it is possible to strongly couple the circumferential modes together which is in disagreement with the harmonic modes as predicted by the orthotropic theory. Also, it is possible for a stiffened cylindrical shell to exhibit two different natural frequencies depending upon whether the shell is excited in a symmetric or antisymmetric mode of vibration.

The results of the mathematical model were compared with the experimental results found in the literature and with the results of a Ritz analysis for simply supported cylinders. These results were found to be in excellent agreement.

No experimental results on the free vibration of longitudinally stiffened cylindrical shells have been found for boundary conditions other than simply supported. Hence no comparison between theory and experiment has been possible. However, a numerical example is presented for a clamped-clamped stiffened cylinder and the results are compared with the experimental results for the same unstiffened shell to illustrate the use of the theory.

Recommendations

Further investigations might be made along the following lines.

A. An extension of the present analysis to the forced vibration case. Perhaps a good starting point would be to consider harmonic forcing functions and simply supported boundary conditions.

B. It is recommended that experimental data be obtained for boundary conditions other than simply supported with which to compare theoretical results.

APPENDIX

APPENDIX A

THIN-WALLED BEAM THEORY

The assumptions made concerning the thin-walled beams are as follows:

(1) The section contour is undeformable--the shape of the cross section does not change in its own plane.

(2) The shear strain at the middle surface of the thin-walled beam is assumed to be zero.

(3) The rotations of the cross section of the beam in its own plane are small.

(4) The thin-walled beams are homogenous and isotropic.

(5) The thin-walled beams are made of Hookean material.

(6) The cross section of the thin-walled beams are constant.

It is these assumptions which enable the equations of motion of a thin-walled beam to be reduced to a set of linear partial differential equations as was demonstrated by V. Z. Vlasov [10].

The following notation is employed (see Figure 15):

0	shear center of thin-walled beam.
C	centroid of the cross section.
ξ	displacement of shear center in y-direction.
η	displacement of shear center in z-direction.
ζ	displacement of the coordinate origin in x-direction.
θ	angle of twist of cross section.

v,w in-plane displacements of a point on section contour along coordinate axes.

u longitudinal displacement of point on stringer profile.

By employing V. Z. Vlasov's assumptions for a thin-walled beam, one can assume displacements of the form:

$$u = \zeta - y(s)\,\frac{\partial\xi}{\partial x} - z(s)\,\frac{\partial\eta}{\partial x} - \omega(s)\,\frac{\partial\theta}{\partial x} \tag{A.1}$$

$$v = \xi - [z(s)-a_z]\theta(x,t) \tag{A.2}$$

$$w = \eta + [y(s)-a_y]\theta(x,t) \tag{A.3}$$

V. Z. Vlasov [10] showed that the strain energy of a thin-walled beam can be shown to be:

$$V_s = \frac{1}{2}\int_0^{\ell} \{E^*A(\zeta')^2 + E^*(\xi'')^2 I_z + E^*(\eta'')^2 I_y + 2E^* I_{yz}\xi''\eta'' +$$

$$E^*(\theta'')^2 C_w + GJ(\theta')^2\}dx \tag{A.4}$$

where the prime (') denotes differentiation with respect to x and with the origin of coordinates at the centroid.

If the origin of coordinates is assumed to be located at the centroid then the kinetic energy of a stringer is given by:

$$T_s = \frac{1}{2}\int_0^{\ell} \{\rho A\left(\dot{\zeta}^2 + \dot{\xi}^2 + \dot{\eta}^2\right) + \rho I_z(\dot{\xi}')^2 + \rho I_y(\dot{\eta}')^2 + 2\rho I_{yz}(\dot{\xi}')(\dot{\eta}')$$

$$+ (\dot{\theta}')^2 C_w + I_p(\dot{\theta})^2 - 2\dot{\eta}\dot{\theta}a_y A + 2\dot{\xi}\dot{\theta}a_z A\}dx \tag{A.5}$$

which includes the effects of rotatory inertia. If $a_y = a_z = 0$ then the shear center coincides with the centroid. If one neglects the rotatory inertia terms then the resulting equations are those used by Gere [18]. Vlasov states that studies in bending vibrations show that for most cases rotatory inertia is indeed negligible.

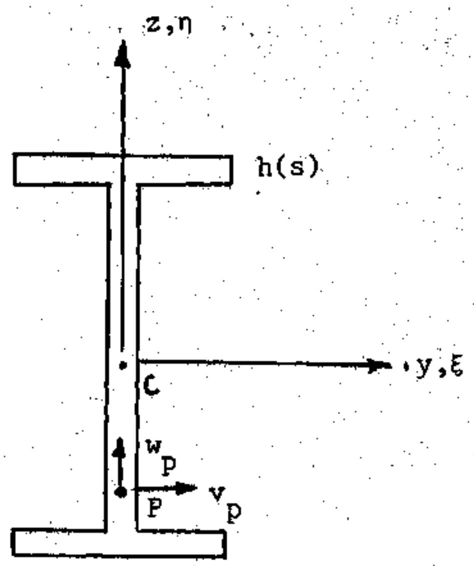


Figure 16. Stringer Displacements and Notation

APPENDIX B

DONNELL'S EQUATIONS OF MOTION

If one neglects the rotatory inertia effect, then the dynamic equations of motion of a cylindrical shell can be written as: (inward w is positive)

$$\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{\phi x}}{\partial \phi} + p_x = \rho h \frac{\partial^2 u}{\partial t^2}, \tag{B.1}$$

$$\frac{1}{a}\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} - \frac{Q_\phi}{a} + p_\phi = \rho h \frac{\partial^2 v}{\partial t^2}, \tag{B.2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{1}{a}\frac{\partial Q_\phi}{\partial \phi} + \frac{N_\phi}{a} + p_z = \rho h \frac{\partial^2 w}{\partial t^2}, \tag{B.3}$$

$$\frac{\partial M_x}{\partial x} + \frac{1}{a}\frac{\partial M_{\phi x}}{\partial \phi} - Q_x = 0, \tag{B.4}$$

$$\frac{\partial M_x}{\partial x} + \frac{1}{a}\frac{\partial M_\phi}{\partial \phi} - Q_\phi = 0, \tag{B.5}$$

$$N_{x\phi} - N_{\phi x} = 0. \tag{B.6}$$

Eliminating Q_ϕ and Q_x results in the following four equations

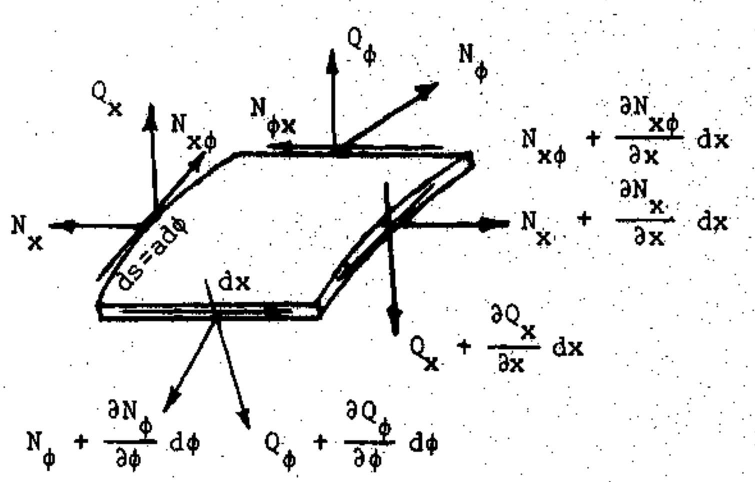


Figure 17(a). Shell Element Showing Stress Resultants

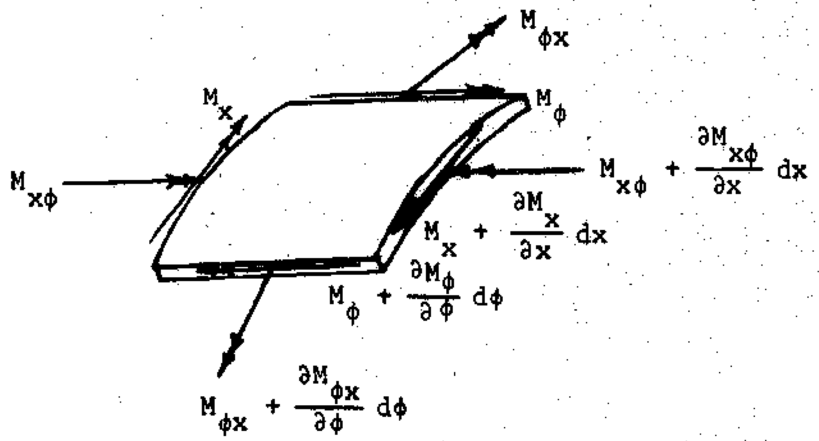


Figure 17(b). Shell Element Showing Stress Couples

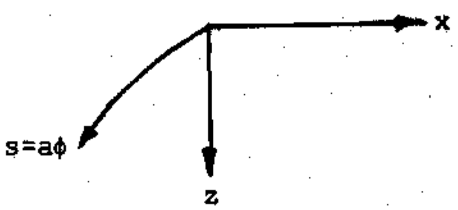


Figure 17(c). Shell Element Showing Coordinate System

$$\frac{\partial N_x}{\partial x} + \frac{1}{a}\frac{\partial N_{x\phi}}{\partial \phi} + p_x = \rho h \frac{\partial^2 u}{\partial t^2}, \qquad (B.7)$$

$$\frac{1}{a}\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} - \frac{1}{a}\frac{\partial M_{x\phi}}{\partial x} - \frac{1}{a^2}\frac{\partial M_\phi}{\partial \phi} + p_\phi = \rho h \frac{\partial^2 v}{\partial t^2}, \qquad (B.8)$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{1}{a}\frac{\partial^2 M_{\phi x}}{\partial x \partial \phi} + \frac{1}{a}\frac{\partial^2 M_{x\phi}}{\partial x \partial \phi} + \frac{1}{a^2}\frac{\partial^2 M_\phi}{\partial \phi^2} + \frac{1}{a} N_\phi + p_z = \rho h \frac{\partial^2 w}{\partial t^2}, \qquad (B.9)$$

$$N_{x\phi} = N_{\phi x}, \qquad (B.10)$$

The stress resultants and stress couples are related to the strains and changes of curvature by:

$$N_{xx} = K[e_{xx} + \nu e_{\phi\phi}], \qquad (B.11)$$

$$N_{\phi\phi} = K[e_{\phi\phi} + \nu e_{xx}], \qquad (B.12)$$

$$N_{x\phi} = N_{\phi x} = Gh\ \gamma_{x\phi}, \qquad (B.13)$$

$$M_{xx} = D[K_x + \nu K_\phi], \qquad (B.14)$$

$$M_{\phi\phi} = D[K_\phi + \nu K_x], \qquad (B.15)$$

$$M_{x\phi} = M_{\phi x} = \frac{Gh^3}{12}\tau, \qquad (B.16)$$

where

$$\{K,D\} = \{h,\frac{h^3}{12}\}E/(1-\nu^2), \tag{B.17}$$

and

$$G = \frac{E}{2(1+\nu)} . \tag{B.18}$$

The strain-displacement relations are given by:

$$e_{xx} = \frac{\partial u}{\partial x} , \tag{B.19}$$

$$e_{\phi\phi} = \frac{1}{a}\frac{\partial v}{\partial \phi} - \frac{w}{a} , \tag{B.20}$$

$$\gamma_{x\phi} = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \phi} . \tag{B.21}$$

The moment-change in curvature relations as advanced by Donnell [20] are:

$$K_x = -\frac{\partial^2 w}{\partial x^2} , \tag{B.22}$$

$$K_\phi = -\frac{1}{a^2}\frac{\partial^2 w}{\partial \phi^2} , \tag{B.23}$$

$$\tau = -\frac{2}{a}\frac{\partial^2 w}{\partial x \partial \phi} . \tag{B.24}$$

By substituting the curvature expressions (Equations B.22, B.23, and B.24) and the strain expressions (Equations B.19, B.20, and B.21) into the expressions for the stress resultants and stress couples and then substituting the latter expressions into Equations (B.7), (B.8) and (B.9) yields:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x\partial\phi} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{(1-\nu^2)}{Eh}p_x = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 u}{\partial t^2}, \qquad (B.25)$$

$$\frac{1+\nu}{2a}\frac{\partial^2 u}{\partial x\partial\phi} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial\phi^2} - \frac{1}{a^2}\frac{\partial w}{\partial\phi}$$

$$+ \frac{(1-\nu^2)}{Eh}p_\phi = \frac{(1-\nu^2)}{E}\frac{\partial^2 v}{\partial t^2}, \qquad (B.26)$$

$$\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial\phi} - \frac{w}{a^2} - \frac{h^2}{12}\nabla^4 w + \frac{(1-\nu^2)}{Eh}p_z = \frac{(1-\nu^2)\rho}{E}\frac{\partial^2 w}{\partial t^2}. \qquad (B.27)$$

Applying $\frac{\partial^2}{\partial x^2}$, $\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}$, and $\frac{\partial^2}{\partial t^2}$ to Equation (B.25), solving in each case for the term involving n, and substituting these expressions into the equation obtained by applying $\frac{1}{a}\frac{\partial^2}{\partial x\partial\phi}$ to Equation (B.26) gives:

$$\nabla^4 u - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1}{a^2}\frac{\partial^3 w}{\partial x\partial\phi^2} = -\frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\left[\frac{(1-\nu^2)}{E}\rho\frac{\partial^2 u}{\partial t^2} + \right.$$

$$\left.\frac{\nu}{a}\frac{\partial w}{\partial x} - \frac{3-\nu}{2}\nabla^2 u - \frac{(1-\nu^2)}{Eh}p_x\right] - \frac{(1-\nu^2)}{Eh}\frac{\partial^2 p_x}{\partial x^2} -$$

$$\frac{(1-\nu^2)}{Eh}\frac{2}{(1-\nu)}\frac{1}{a^2}\frac{\partial^2 p_x}{\partial\phi^2} + \frac{(1-\nu^2)}{Eh}\frac{(1+\nu)}{(1-\nu)}\frac{1}{a}\frac{\partial^2 p}{\partial x\partial\phi}. \qquad (B.28)$$

Similarly, applying $\frac{\partial^2}{\partial x^2}$, $\frac{1}{a^2}$ $\frac{\partial^2}{\partial\phi^2}$, and $\frac{\partial^2}{\partial t^2}$ to Equation (B.26), solving in each case for the term involving u, and substituting these in Equation (B.25), after applying $\frac{1}{a}\frac{\partial^2}{\partial x\partial\phi}$ to it, gives:

$$\nabla^4 v - \frac{2+\nu}{a^2}\frac{\partial^3 w}{\partial x^2 \partial\phi} - \frac{1}{a^4}\frac{\partial^3 w}{\partial\phi^3} = -\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left[\frac{(1-\nu^2)\rho}{E}\frac{\partial^2 v}{\partial t^2} - \right.$$

$$\left. - \frac{3-\nu}{2}\nabla^2 v + \frac{1}{a^2}\frac{\partial w}{\partial\phi} - \frac{(1-\nu^2)}{Eh}p_\phi\right] - \frac{(1-\nu^2)}{Eh}\frac{1}{a^2}\frac{\partial^2 p_\phi}{\partial\phi^2} -$$

$$\frac{(1-\nu^2)}{Eh}\frac{2}{(1-\nu)}\frac{\partial^2 p_\phi}{\partial x^2} + \frac{(1-\nu^2)}{Eh}\frac{(1+\nu)}{(1-\nu)}\frac{1}{a}\frac{\partial^2 p_x}{\partial x\partial\phi} . \qquad (B.29)$$

The third equation is obtained by applying $\frac{\nu}{a}\frac{\partial}{\partial x}$ to Equation (B.28) and $\frac{1}{a^2}\frac{\partial}{\partial\phi}$ to Equation (B.29) and adding the results gives:

$$\nabla^4\left(\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial\phi}\right) - \frac{\nu^2}{a^2}\frac{\partial^4 w}{\partial x^4} - \frac{2}{a^4}\frac{\partial^4 w}{\partial x^2\partial\phi^2} - \frac{1}{a^6}\frac{\partial^4 w}{\partial\phi^4} =$$

$$- \frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\quad\frac{(1-\nu^2)\rho}{E}\frac{\partial^2}{\partial t^2}\left(\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial\phi}\right) -$$

$$- \frac{3-\nu}{2}\nabla^2\left(\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial\phi}\right) + \frac{\nu^2}{a^2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4}\frac{\partial^2 w}{\partial\phi^2}\Bigg] -$$

$$- \frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\left[-\frac{\nu(1-\nu^2)}{aEh}\frac{\partial p_x}{\partial x} - \frac{(1-\nu^2)}{Eha^2}\frac{\partial p_\phi}{\partial\phi}\right]$$

$$- \frac{(1-\nu^2)}{aEh}\frac{\partial^3 p_x}{\partial x^3} - \frac{2\nu}{a^3(1-\nu)}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_x}{\partial x\partial\phi^2} -$$

$$- \frac{1}{a^4}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial\phi^3} - \frac{2}{a^2(1-\nu)}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial x^2\partial\phi}$$

$$+ \frac{1}{a^3}\frac{(1+\nu)}{(1-\nu)}\left(\frac{1-\nu^2}{Eh}\right)\frac{\partial^3 p_x}{\partial x\partial\phi^2} + \frac{\nu}{a^2}\frac{(1+\nu)}{(1-\nu)}\left(\frac{1-\nu^2}{Eh}\right)\frac{\partial^3 p_\phi}{\partial x^2\partial\phi} . \qquad (B.30)$$

The u and v terms in Equation (B.30) appear together in the expression:

$$\frac{\nu}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial \phi}$$

If this expression is solved from Equation (B.27) in terms of w and its derivatives and the result substituted into Equation (B.30), we have:

$$\frac{h^2}{12}\nabla^8 w + \frac{1-\nu^2}{a^2}\frac{\partial^4 w}{\partial x^4} - \frac{1-\nu^2}{Eh}\nabla^4 p_z = -\frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}\Bigg[\{\frac{1-\nu^2}{E}\rho\frac{\partial^2}{\partial t^2}$$

$$-\frac{3-\nu}{2}\nabla^2\}\{\frac{\rho(1-\nu^2)}{E}\frac{\partial^2 w}{\partial t^2} + \frac{w}{a^2} + \frac{h^2}{12}\nabla^4 w - \frac{(1-\nu^2)}{Eh}p_z\} +$$

$$\frac{(1-\nu)}{2}\nabla^4 w + \frac{\nu^2}{a^2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4}\frac{\partial^2 w}{\partial \phi^2}\Bigg] - \frac{2(1+\nu)}{E}\rho\frac{\partial^2}{\partial t^2}$$

$$\left[-\frac{\nu(1-\nu)}{aEh}\frac{\partial p_x}{\partial x} - \frac{(1-\nu^2)}{Eha^2}\frac{\partial p_\phi}{\partial \phi}\right] - \frac{\nu(1-\nu^2)}{aEh}\frac{\partial^3 p_x}{\partial x^3} -$$

$$\frac{2\nu(1-\nu^2)}{a^3(1-\nu)Eh}\frac{\partial^3 p_x}{\partial x\partial\phi^2} - \frac{1}{a^4}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial \phi^3} - \frac{2(1-\nu^2)}{a^2(1-\nu)Eh}\frac{\partial^3 p_x}{\partial x^2\partial\phi} +$$

$$\frac{1}{a^3}\frac{(1+\nu)}{(1-\nu)}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_x}{\partial x\partial\phi^2} + \frac{\nu}{a}\frac{(1+\nu)}{(1-\nu)}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial x^2\partial\phi}. \qquad \text{(B.31)}$$

After some simplification, we have:

$$\frac{h^2}{12}\nabla^8 w + \frac{(1-\nu^2)}{a^2}\frac{\partial^4 w}{\partial x^4} - \frac{(1-\nu^2)}{Eh}\nabla^4 p_z =$$

$$- \frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\Bigg[\left\{\frac{(1-\nu^2)\rho}{E}\frac{\partial^2}{\partial t^2} - \frac{(3-\nu)}{2}\nabla^2\right\}$$

$$\left\{\frac{\rho(1-\nu^2)}{E}\frac{\partial^2 w}{\partial t^2} + \frac{w}{a^2} + \frac{h^2}{12}\nabla^4_w - \frac{(1-\nu^2)}{Eh}p_z\right\}$$

$$+ \frac{(1-\nu)}{2}\nabla^4_w + \frac{\nu^2}{a^2}\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4}\frac{\partial^2 w}{\partial \phi^2}\Bigg]$$

$$- \frac{2(1+\nu)\rho}{E}\frac{\partial^2}{\partial t^2}\left[-\frac{\nu(1-\nu)}{aEh}\frac{\partial p_x}{\partial x} - \frac{(1-\nu^2}{Eha^2}\frac{\partial p_\phi}{\partial \phi}\right]$$

$$- \frac{\nu(1-\nu^2)}{aEh}\frac{\partial^3 p_x}{\partial x^3} + \frac{(1-\nu^2)(1-\nu)}{a^3(1-\nu)Eh}\frac{\partial^3 p_x}{\partial x \partial \phi^2}$$

$$- \frac{1}{a^4}\frac{(1-\nu^2)}{Eh}\frac{\partial^3 p_\phi}{\partial \phi^3} - \frac{(2+\nu)(1-\nu^2)(1-\nu^2)}{a^2(1-\nu)Eh}\frac{\partial^3 p_\phi}{\partial x^2 \partial \phi} . \quad (B.32)$$

APPENDIX C

MATRIX ELEMENTS IN RITZ ANALYSIS

This appendix contains expressions for the matrix elements appearing in the submatrices of the mass matrix of Equation (3.15) and the stiffness matrix given by Equation (3.14). The axial mode number (m) is considered as being given. The elements of the submatrices of the stiffness matrix are given as follows:

$$k_{11}(j,n) = \delta_{jn}[a^2\alpha_m^2 + \frac{1}{2}(1-\nu)_j^2(1+\beta)]$$

$$+ a^2\alpha_m^2 \sum_{i=1}^{K} D_{1i} \cos n\phi_i \cos j\phi_i, \qquad (C.1)$$

$$k_{12}(j,n) = -\delta_{jn}[a_j \frac{1}{2}(1+\nu)\alpha_m] + a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i}(y_i) \sin n\phi_i \cos j\phi_i, \qquad (C.2)$$

$$k_{13}(j,n) = \delta_{jn}[- a \; _m - \beta a^3\alpha_m + \frac{(1-\nu)}{2} \alpha_{mj}\beta]$$

$$+ a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i} Z_i \cos n\phi_i \cos j\phi_i, \qquad (C.3)$$

$$k_{14}(j,n) = a^2\alpha_m^2 \sum_{i=1}^{K} D_{1i} \sin n\phi_i \cos j\phi_i, \qquad (C.4)$$

$$k_{15}(j,n) - a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i}(y_i)\cos n\phi_i \cos j\phi_i, \qquad (C.5)$$

$$k_{16}(j,n) = +a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i}Z_i \sin n\phi_i \cos j\phi_i) \qquad (C.6)$$

$$k_{22}(j,n) = \delta_{jn}\left[j^2 + \frac{1}{2}(1-\nu)a^2\alpha_m^2 + \beta\{\frac{3}{2}(1-\nu)a^2\alpha_m^2\}\right]$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}y_i^2 \sin n\phi_i \sin j\phi_i$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i} I_{zi}/A_i \sin(n\phi_i)\sin(j\phi_i) \qquad (C.7)$$

$$k_{23}(j,n) = \delta_{jn}(j)[1 + \beta(\nu a^2\alpha_m + \frac{3}{2}(1-\nu)a^2\alpha_m^2)]$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}y_iZ_i \cos(n\phi_i)\sin(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i} I_{yzi}/A_i \cos n\phi_i \sin j\phi_i, \qquad (C.8)$$

$$k_{24}(j,n) = +a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i}(y_i)\sin(n\phi_i)\sin(j\phi_i), \qquad (C.9)$$

$$k_{25}(j,n) = -a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}(y_i^2)\cos(n\phi_i)\sin(j\phi_i)$$

$$- a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{zi}/A_i(\cos n\phi_i)(\sin j\phi\ ), \qquad (C.10)$$

$$k_{26}(j,n) = a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}(y_i)Z_i)\sin(n\phi_i)\cos(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_1 i\ I_{yzi}/A_i\ \sin(n\phi_i)\sin(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_1 i\ I_{zi}/A_i\left(\frac{Z_i}{a}\right)(n)\sin(j\phi_i)\cos(n\phi_i), \qquad (C.11)$$

$$k_{33}(j,n) = \delta_{jn}[1 + \beta(1+n^4 - 2n^2 + a^4\alpha_m^4 + 2a^2\alpha_n^2 n^2)]$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_1 i(Z_i^2)\cos n\phi_i\ \cos j\phi_i\ +$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{yi}/A_i\ \cos(n\phi_i)\cos(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{3i}(j)(n)\sin(n\phi_i)\sin(j\phi_i)$$

$$+ \alpha_m^2 \sum_{i=1}^{K} D_{2i}(j)(n)\sin(n\phi_i)\sin(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{yzi}/A_i \left(\frac{Z_i}{a}\right)\cos(j\phi_i)\sin(n\phi_i)$$

$$- a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{zi}/A_i \left(\frac{Z_i}{a}\right)(j)(n)\sin(n\phi_i)\sin(j\phi_i), \qquad (C.12)$$

$$k_{34}(j,n) = +a^2\alpha_m^3 \sum_{i=1}^{K} D_{1i}Z_i\sin(n\phi_i)\cos(j\phi_i), \qquad (C.13)$$

$$k_{35}(j,n) = a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}(y_i)Z_i)\cos(n\phi_i)\cos(j\phi_i)$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{yzi}/A_i \sin n\phi_i \sin j\phi_i$$

$$+a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{zi}/A_i \left(\frac{Z_i}{a}\right)(n)\sin(n\phi_i)\cos(j\phi_i), \qquad (C.14)$$

$$k_{36}(j,n) = a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}(Z_i^2)\sin n\phi_i \cos j\phi_i$$

$$+ a^2\alpha_m^4 \sum_{i=1}^{K} D_{1i}\ I_{yi}/A_i \sin n\phi_i \cos j\phi_i$$

$$- a^2\alpha_m^4 \sum_{i=1}^{K} D_{3i}(j)(n)\cos n\phi_i \sin j\phi_i$$

$$- a^2\alpha_m^2 \sum_{i=1}^{K} D_{2i}(j)(n)\sin n\phi_i \cos j\phi_i. \qquad (C.15)$$

The elements of the submatrices of the mass matrix are given by the following expressions:

$$m_{11}(j,n) = \delta_{jn} + \sum_{i=1}^{K} D_{4i} \cos(n\phi_i)\cos(j\phi_i), \tag{C.16}$$

$$m_{12}(j,n) = \alpha_m \sum_{i=1}^{K} D_{4i}(y_i)\sin(n\phi_i)\cos(j\phi_i), \tag{C.17}$$

$$m_{13}(j,n) = +\alpha_m \sum_{i=1}^{K} D_{4i}(Z_i)\cos(n\phi_i)\cos(j\phi_i), \tag{C.18}$$

$$m_{14}(j,n) = \sum_{i=1}^{K} D_{4i}\sin(n\phi_i)\cos(j\phi_i), \tag{C.19}$$

$$m_{15}(j,n) = \sum_{i=1}^{K} D_{4i}(y_i)\cos(n\phi_i)\cos(j\phi_i), \tag{C.20}$$

$$m_{16}(j,n) = +\alpha_m \sum_{i=1}^{K} D_{4i}(Z_i)\sin(n\phi_i)\cos(j\phi_i) \tag{C.21}$$

$$m_{22}(j,n) = \delta_{jn}(1+\beta) + \sum_{i=1}^{K} D_{4i}[1 + (y_i^2) + \alpha_m^2 I_{zi}/A_i]\sin(n\phi_i)\sin(j\phi_i), \tag{C.22}$$

$$m_{23}(j,n) = \delta_{jn}(2\beta_j) - \sum_{i=1}^{K} D_{4i}\left(\frac{a_{zi}}{a}\right)(n)\sin(n\phi_i)\sin(j\phi_i) +$$

$$\alpha_m^2 \sum_{i=1}^{K} D_{4i}(I_{yzi}/A_i)\cos(n\phi_i)\sin(j\phi_i) -$$

$$\sum_{i=1}^{K} D_{4i}\left(\frac{Z_i}{a}\right)(n)\sin(n\phi_i)\sin(j\phi_i)$$

$$- \alpha_m^2 \sum_{i=1}^{K} D_{4i}(y_i)(Z_i)\cos(n\phi_i)\sin(j\phi_i), \tag{C.23}$$

$$m_{24}(j,n) = +\alpha_m \sum_{i=1}^{K} D_{4i}(y_i)\sin(n\phi_i)\sin(j\phi_i), \qquad (C.24)$$

$$m_{25}(j,n) = -\sum_{i=1}^{K} D_{4i}[1 + (y_i^2) + I_{zi}\ {}_m^2/A_i]\cos n\phi_i \sin j\phi_i, \qquad (C.25)$$

$$m_{26}(j,n) = \sum_{i=1}^{K} D_{4i}[\alpha_m^2\ I_{yzi}/A_i + \alpha_m^2(y_i)(Z_i)]\sin(n\phi_i)\sin(j\phi_i) +$$

$$\sum_{i=1}^{K} \{D_{4i}\left(\frac{a_{zi}}{a}\right)(n) + D_{4i}\left(\frac{Z_i}{a}\right)(n)\} \cos(n\phi_j)\sin(j\phi_i), \qquad (C.26)$$

$$m_{33}(j,n) = \delta_{jn}[1 + \beta(a^2\alpha_m^2+j^2)] + \sum_{i=1}^{K} D_{4l}[1 + Z_i^2 + \alpha_m^2\ I_{yi}/A_i]$$

$$\cos(n\phi_i)\cos(j\phi_i) + \sum_{i=1}^{K} D_{4i}(j)(n)\left[\left(\frac{Z_i}{a}\right)^2 + \left(\frac{y_i}{a}\right)^2 +\right.$$

$$\left. I_{pi}/a^2A_i\right]\sin(n\phi_i)\sin(j\phi_i) + \sum_{i=1}^{K} D_{4i}\ \frac{(y_i)}{a}\ (j)\cos(n\phi_i)\sin(j\phi_i)$$

$$+ \sum_{i=1}^{K} D_{4i}\left(\frac{y_i}{a}\right)(n)\sin(n\phi_i)\cos(j\phi_i)$$

$$+ \sum_{i=1}^{K} D_{4i}\left(\frac{a_{yi}}{a}\right)(n)\cos(j\phi_i)\sin(n\phi_i), \qquad (C.27)$$

$$m_{34}(j,n) = \alpha_m + \sum_{i=1}^{K} D_{4i}(Z_i)\sin(n\phi_i)\cos(j\phi_i), \qquad (C.28)$$

$$m_{35}(j,n) = + \sum_{i=1}^{K} D_{4i}\left(\frac{Z_i}{a}\right)(n)\cos(n\phi_i)\sin(j\phi_i)$$

$$- \alpha_m^2 \sum_{i=1}^{K} D_{4i}(y_i)Z_i)\sin(n\phi_i)\sin(j\phi_i)$$

$$+ \sum_{i=1}^{K} D_{4i}\left(\frac{a_{zi}}{a}\right)(n)\cos(n\phi_i)\sin(j\phi_i)$$

$$- \alpha_m^2 \sum_{i=1}^{K} D_{4i}\ \frac{I_{yzi}}{A_i} \cos(n\phi_i)\cos(j\phi_i), \tag{C.29}$$

$$m_{36}(j,n) = \alpha_m^2 \sum_{i=1}^{K} D_{4i}(Z_i^2)\sin(n\phi_i)\cos(j\phi_i)$$

$$- \sum_{i=1}^{K} D_{4i}\left(\frac{Z_i}{a}\right)^2 (j)(n)\cos(n\phi_i)\sin(j\phi_i)$$

$$+ \sum_{i=1}^{K} D_{4i}\sin(n\phi_i)\cos(j\phi_i)$$

$$- \sum_{i=1}^{K} D_{4i}\left(\frac{y_i}{a}\right)(n)\cos(n\phi_i)\cos(j\phi_i)$$

$$+ \sum_{i=1}^{K} D_{4i}\left(\frac{y_i}{a}\right)(j)\sin(n\phi_i)\sin(j\phi_i) - \sum_{i=1}^{K} D_{4i}\left(\frac{y_i}{a}\right)^2 (j)(n)\cos(n\phi_i)\sin(j\phi_i)$$

$$- \sum_{i=1}^{K} D_{4i}\left(\frac{I_{pi}}{A_i a^2}\right)(j)(n)\cos(n\phi_i)\sin(j\phi_i) - \sum_{i=1}^{K} D_{4l}\left(\frac{a_{yi}}{a}\right)(n)\cos(n\phi_i)\cos(n\phi_i) \tag{C.30}$$

The elements of the submatrices (m_{11}^a), $m_{12}^a)$, (m_{13}^a), (m_{22}^a), (m_{23}^a), and (m_{33}^a) which represent the antisymmetric circumferential modes of vibration may be obtained from the elements of the submatrices (m_{11}), (m_{12}), (m_{13}), (m_{22}), (m_{23}) and (m_{33}) by interchanging $\sin(n\phi_i)$ and $\cos(n\phi_i)$ and y_i with $-h_i$. The constants appearing in the expressions for the mass and stiffness submatrices are given by:

$$D_{1i} = \frac{E_i^* A_i (1-\nu^2)}{E\pi a h}, \tag{C.31}$$

$$D_{2i} = \frac{(GJ)_i (1-\nu^2)}{E\pi a^3 h}, \tag{C.32}$$

$$D_{3i} = \frac{C_{wi} (1-\nu^2)}{E\pi a^3 h}, \tag{C.33}$$

$$D_{4i} = \frac{\rho_i A_i}{p\pi a h}, \tag{C.34}$$

where $(GJ)_i$ is the torsional stiffness and C_{wi} is the warping constant for the ith stringer. When Donnell's shell theory is used the expressions for the shell theory in Equations (C.1) to (C.15) are changed to:

$$k_{11}(j,n) = \delta_{jn}\{a^2\alpha_m^2 + \frac{1}{2}(1-\nu)_j^2\}, \tag{C.35}$$

$$k_{12}(j,n) = -\delta_{nj}\{a_j \frac{1}{2}(1+\nu)\}\alpha_m, \tag{C.36}$$

$$k_{13}(j,n) = \delta_{jn}\{-a\nu\alpha_m\}, \tag{C.37}$$

$$k_{22}(j,n) = \delta_{jn}\{j^2 + \frac{1}{2}(1-\nu)a^2\alpha_m^2\}, \tag{C.38}$$

$$k_{23}(j,n) = \delta_{jn}(j), \tag{C.39}$$

$$k_{33}(j,n) = \delta_{jn}[1 + \beta\{n^4+a^4\alpha_m^4+2a^2n^2\alpha_m^2\}] \tag{C.40}$$

where $\beta = h^2/12a^2$ and the beta terms represent the bending energy of the shell.

APPENDIX D

MATRIX ELEMENTS FOR CLAMPED-CLAMPED SUPPORTS

The nonzero elements of the coefficient matrix $[\bar{D}]$ for clamped-clamped boundary are given as follows:

$$d_{11} = \bar{\alpha}_{1n},$$

$$d_{12} = \bar{\alpha}_{2n},$$

$$d_{13} = \bar{\alpha}_{3n},$$

$$d_{14} = \bar{\alpha}_{4n},$$

$$d_{15} = \bar{\alpha}_{5n},$$

$$d_{16} = \bar{\alpha}_{6n},$$

$$d_{17} = \bar{\alpha}_{7n},$$

$$d_{18} = \bar{\alpha}_{8n},$$

$$d_{21} = \bar{\beta}_{1n},$$

$$d_{22} = \bar{\beta}_{2n},$$

$$d_{23} = \bar{\beta}_{3n},$$

$$d_{24} = \bar{\beta}_{4n},$$

$$d_{25} = \bar{\beta}_{5n},$$

$$d_{26} = \bar{\beta}_{6n},$$

$$d_{27} = \bar{\beta}_{7n},$$

$$d_{28} = \bar{\beta}_{8n},$$

$$d_{31} = 1,$$

$$d_{32} = 0,$$

$$d_{33} = 1,$$

$$d_{34} = 0,$$

$$d_{35} = 1,$$

$$d_{36} = 0,$$

$$d_{37} = 1,$$

$$d_{38} = 0,$$

$$d_{41} = \psi_{1n},$$

$$d_{42} = \theta_{1n},$$

$$d_{43} = \psi_{2n},$$

$$d_{44} = \theta_{2n},$$

$$d_{45} = \theta_{3n},$$

$$d_{46} = \psi_{3n},$$

$$d_{47} = \psi_{4n},$$

$$d_{48} = \theta_{4n},$$

$$d_{51} = e^{\psi_{1n} \frac{\ell}{a}}(\bar{\alpha}_{1n} \cos \theta_{1n} \ \ell/a - \bar{\alpha}_{2n} \sin \theta_{1n} \ \ell/a).$$

$$d_{52} = e^{\psi_{1n} \frac{\ell}{a}}(\bar{\alpha}_{2n} \cos \theta_{1n} \ \ell/a + \bar{\alpha}_{1n} \sin \theta_{1n} \ \ell/a),$$

$$d_{53} = e^{\psi_{2n} \frac{\ell}{a}}(\bar{\alpha}_{3n} \cos \theta_{2n} \ell/a - \bar{\alpha}_{4n} \sin \theta_{2n} \ell/a),$$

$$d_{54} = e^{\psi_{2n} \frac{\ell}{a}}(\bar{\alpha}_{4n} \cos \theta_{2n} \ell/a + \bar{\alpha}_{3n} \sin \theta_{2n} \ell/a),$$

$$d_{55} = e^{\psi_{3n} \frac{\ell}{a}}(\bar{\alpha}_{5n} \cos \theta_{3n} \ell/a - \bar{\alpha}_{6n} \sin \theta_{3n} \ell/a),$$

$$d_{56} = e^{\psi_{3n} \frac{\ell}{a}}(\bar{\alpha}_{6n} \cos \theta_{3n} \ell/a + \bar{\alpha}_{5n} \sin \theta_{3n} \ell/a),$$

$$d_{57} = e^{\psi_{4n} \frac{\ell}{a}}(\bar{\alpha}_{7n} \cos \theta_{4n} \ell/a - \bar{\alpha}_{8n} \sin \theta_{4n} \ell/a),$$

$$d_{58} = e^{\psi_{4n} \frac{\ell}{a}}(\bar{\alpha}_{8n} \cos \theta_{4n} \ell/a + \bar{\alpha}_{8n} \sin \theta_{4n} \ell/a),$$

$$d_{61} = e^{\psi_{1n} \frac{\ell}{a}}(\bar{\beta}_{1n} \cos \theta_{1n} \ell/a - \bar{\beta}_{2n} \sin \theta_{1n} \ell/a),$$

$$d_{62} = e^{\psi_{1n} \frac{\ell}{a}}(\bar{\beta}_{2n} \cos \theta_{1n} \ell/a + \bar{\alpha}_{1n} \sin \theta_{1n} \ell/a),$$

$$d_{63} = e^{\psi_{2n} \frac{\ell}{a}}(\bar{\beta}_{3n} \cos \theta_{2n} \ell/a - \bar{\beta}_{4n} \sin \theta_{2n} \ell/a),$$

$$d_{64} = e^{\psi_{2n} \frac{\ell}{a}}(\bar{\beta}_{4n} \cos \theta_{2n} \ell/a + \bar{\beta}_{3n} \sin \theta_{2n} \ell/a),$$

$$d_{65} = e^{\psi_{3n} \frac{\ell}{a}}(\bar{\beta}_{5n} \cos \theta_{3n} \ell/a - \bar{\beta}_{6n} \sin \theta_{3n} \ell/a),$$

$$d_{66} = e^{\psi_{3n} \frac{\ell}{a}}(\bar{\beta}_{6n} \cos \theta_{3n} \ell/a + \bar{\beta}_{5n} \sin \theta_{3n} \ell/a),$$

$$d_{67} = e^{\psi_{4n} \frac{\ell}{a}}(\bar{\beta}_{7n} \cos \theta_{4n} \ell/a - \bar{\beta}_{8n} \sin \theta_{4n} \ell/a),$$

$$d_{68} = e^{\psi_{4n} \frac{\ell}{a}}(\bar{\beta}_{8n} \cos \theta_{4n} \ell/a + \bar{\beta}_{7n} \sin \theta_{4n} \ell/a),$$

$$d_{71} = e^{\psi_{1n} \frac{\ell}{a}} \cos \theta_{1n} \frac{\ell}{a} ,$$

$$d_{72} = e^{\psi_{1n} \frac{\ell}{a}} \sin \theta_{1n} \frac{\ell}{a} ,$$

$$d_{73} = e^{\psi_{2n} \frac{\ell}{a}} \cos \theta_{2n} \frac{\ell}{a} ,$$

$$d_{74} = e^{\psi_{2n} \frac{\ell}{a}} \sin \theta_{2n} \frac{\ell}{a} ,$$

$$d_{75} = e^{\psi_{3n} \frac{\ell}{a}} \cos \theta_{3n} \frac{\ell}{a} ,$$

$$d_{76} = e^{\psi_{3n} \frac{\ell}{a}} \sin \theta_{3n} \frac{\ell}{a} ,$$

$$d_{77} = e^{\psi_{4n} \frac{\ell}{a}} \cos \theta_{4n} \frac{\ell}{a} ,$$

$$d_{78} = e^{\psi_{4n} \frac{\ell}{a}} \sin \theta_{4n} \frac{\ell}{a} ,$$

$$d_{81} = \psi_{1n}\, e^{\psi_{1n}\frac{\ell}{a}} \cos \theta_{1n}\frac{\ell}{a} - \theta_{1n}\, e^{\psi_{1n}\frac{\ell}{a}} \sin \theta_{1n}\frac{\ell}{a},$$

$$d_{82} = \psi_{1n}\, e^{\psi_{1n}\frac{\ell}{a}} \sin \theta_{1n}\frac{\ell}{a} + \theta_{1n}\, e^{\psi_{1n}\frac{\ell}{a}} \cos \theta_{1n}\frac{\ell}{a},$$

$$d_{83} = \psi_{2n}\, e^{\psi_{2n}\frac{\ell}{a}} \cos \theta_{2n}\frac{\ell}{a} - \theta_{2n}\, e^{\psi_{2n}\frac{\ell}{a}} \sin \theta_{2n}\frac{\ell}{a},$$

$$d_{84} = \psi_{2n}\, e^{\psi_{2n}\frac{\ell}{a}} \sin \theta_{2n}\frac{\ell}{a} + \theta_{2n}\, e^{\psi_{2n}\frac{\ell}{a}} \cos \theta_{2n}\frac{\ell}{a},$$

$$d_{85} = \psi_{3n}\, e^{\psi_{3n}\frac{\ell}{a}} \cos \theta_{3n}\frac{\ell}{a} - \theta_{3n}\, e^{\psi_{3n}\frac{\ell}{a}} \sin \theta_{3n}\frac{\ell}{a},$$

$$d_{86} = \psi_{3n}\, e^{\psi_{3n}\frac{\ell}{a}} \sin \theta_{3n}\frac{\ell}{a} + \theta_{3n}\, e^{\psi_{3n}\frac{\ell}{a}} \cos \theta_{3n}\frac{\ell}{a},$$

$$d_{87} = \psi_{4n}\, e^{\psi_{4n}\frac{\ell}{a}} \cos \theta_{4n}\frac{\ell}{a} - \theta_{4n}\, e^{\psi_{4n}\frac{\ell}{a}} \sin \theta_{4n}\frac{\ell}{a},$$

$$d_{87} = \psi_{4n}\, e^{\psi_{4n}\frac{\ell}{a}} \sin \theta_{4n}\frac{\ell}{a} + \theta_{4n}\, e^{\psi_{4n}\frac{\ell}{a}} \cos \theta_{4n}\frac{\ell}{a}.$$

APPENDIX E

ALGEBRAIC EXPRESSIONS IN COMPLEMENTARY SOLUTION

This appendix contains expressions for the α's and β's which occur in the complementary solutions to Donnell's equations. These are given as follows:

$$\bar{\beta}_{1n} = \frac{Y_{11}Y_{13} - Y_{12}Y_{14}}{Y_{13}^2 + Y_{14}^2}, \tag{E.1}$$

$$\bar{\beta}_{2n} = \frac{Y_{12}Y_{13} + Y_{11}Y_{14}}{Y_{13}^2 + Y_{14}^2}, \tag{E.2}$$

$$\bar{\beta}_{3n} = \frac{Y_{21}Y_{23} - Y_{22}Y_{24}}{Y_{23}^2 + Y_{24}^2}, \tag{E.3}$$

$$\bar{\beta}_{4n} = \frac{Y_{22}Y_{23} + Y_{21}Y_{24}}{Y_{23}^2 + Y_{24}^2}, \tag{E.4}$$

$$\bar{\beta}_{5n} = \frac{Y_{31}Y_{33} - Y_{32}Y_{34}}{Y_{33}^2 + Y_{34}^2}, \tag{E.5}$$

$$\bar{\beta}_{6n} = \frac{Y_{32}Y_{33} + Y_{31}Y_{34}}{Y_{33}^2 + Y_{34}^2}, \tag{E.6}$$

$$\bar{\beta}_{7n} = \frac{Y_{41}Y_{43} - Y_{42}Y_{44}}{Y_{43}^2 + Y_{44}^2} , \tag{E.7}$$

$$\bar{\beta}_{8n} = \frac{Y_{42}Y_{43} + Y_{41}Y_{44}}{Y_{43}^2 + Y_{44}^2} , \tag{E.8}$$

$$\bar{\alpha}_{1n} = \frac{X_{11}Y_{13} + X_{12}Y_{14}}{Y_{13}^2 + Y_{14}^2} , \tag{E.9}$$

$$\bar{\alpha}_{2n} = \frac{X_{12}Y_{13} - X_{11}Y_{14}}{Y_{13}^2 + Y_{14}^2} , \tag{E.10}$$

$$\bar{\alpha}_{3n} = \frac{X_{21}Y_{23} + X_{22}Y_{24}}{Y_{23}^2 + Y_{24}^2} , \tag{E.11}$$

$$\bar{\alpha}_{4n} = \frac{X_{22}Y_{23} - X_{21}Y_{24}}{Y_{23}^2 + Y_{24}^2} , \tag{E.12}$$

$$\bar{\alpha}_{5n} = \frac{X_{31}Y_{33} + X_{32}Y_{34}}{Y_{33}^2 + Y_{34}^2} , \tag{E.13}$$

$$\bar{\alpha}_{6n} = \frac{X_{32}Y_{33} - X_{31}Y_{34}}{Y_{33}^2 + Y_{34}^2} , \tag{E.14}$$

$$\bar{\alpha}_{7n} = \frac{X_{41}Y_{43} + X_{42}Y_{44}}{Y_{43}^2 + Y_{44}^2} , \tag{E.15}$$

$$\bar{\alpha}_{8n} = \frac{X_{42}Y_{43} - X_{41}Y_{44}}{Y_{43}^2 + Y_{44}^2}, \tag{E.16}$$

where

$$Y_{11} = n\{n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{1n}^2-\theta_{1n}^2)\}, \tag{E.17}$$

$$Y_{12} = \{2n(2+\nu)(\psi_{1n}\theta_{1n})\}, \tag{E.18}$$

$$Y_{13} = \{\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)\Omega n^2}{(1-\nu)} + n^4 + (\psi_{1n}^2-\theta_{1n}^2)\ \{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2 + 1\}$$

$$- 4\psi_{1n}^2\theta_{1n}^2 + (\psi_{1n}^2-\theta_{1n}^2)^2\}, \tag{E.19}$$

$$Y_{14} = \{4\psi_{1n}\theta_{1n}(\psi_{1n}^2-\theta_{1n}^2) + \{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2\}\ (2\psi_{1n}\theta_{1n})\ , \tag{E.20}$$

$$Y_{21} = n\{n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{2n}^2-\theta_{2n}^2)\}, \tag{E.21}$$

$$Y_{22} = \{2n(2+\nu)(\psi_{2n}\theta_{2n})\}, \tag{E.22}$$

$$Y_{23} = \{\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)\Omega^2 n^2}{(1-\nu)} + n^4 + (\psi_{2n}^2-\theta_{2n}^2)\ \{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2 + 1\}$$

$$- 4\psi_{2n}^2\theta_{2n}^2 + (\psi_{2n}^2-\theta_{2n}^2)^2\}, \tag{E.23}$$

$$Y_{24} = \{4\psi_{2n}\theta_{2n}(\psi_{2n}^2-\theta_{2n}^2) + \{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2\}\ (2\psi_{2n}\theta_{2n})\}, \tag{E.24}$$

$$Y_{31} = n\{n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{3n}^2 - \theta_{3n}^2)\}, \tag{E.25}$$

$$Y_{32} = \{2n(2+\nu)(\psi_{3n}\theta_{3n})\}, \tag{E.26}$$

$$Y_{33} = \{\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)\Omega^2 n^2}{(1-\nu)} + n^4 + (\psi_{3n}^2 - \theta_{3n}^2)$$

$$\{\frac{(3-\nu)\Omega}{(1-\nu)} - 2_n^2 + 1\} - 4\psi_{3n}^2\theta_{3n}^2 + (\psi_{3n}^2 - \theta_{3n}^2)^2\}, \tag{E.27}$$

$$Y_{34} = \{4\psi_{3n}\theta_{3n}(\psi_{3n}^2 - \theta_{3n}^2) + \{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2\}\ (2\psi_{3n}\theta_{3n})\}, \tag{E.28}$$

$$Y_{41} = n\{n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{4n}^2 - \theta_{4n}^2)\}, \tag{E.29}$$

$$Y_{42} = \{2n(2+\nu)\ (\psi_{4n}\theta_{4n})\}, \tag{E.30}$$

$$Y_{43} = \{\frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)\Omega^2 n^2}{(1-\nu)} + n^4 + (\psi_{4n}^2 - \theta_{4n}^2)$$

$$\{\frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2 + 1\} -$$

$$4\psi_{4n}^2\theta_{4n}^2 + (\psi_{4n}^2 - \theta_{4n}^2)^2\}, \tag{E.31}$$

$$X_{11} = \left[(\psi_{1n})\ \{\frac{2\nu\Omega}{(1-\nu^2)} + n^2\} + \nu\{(\psi_{1n})(\psi_{1n}^2 - \theta_{1n}^2) - 2\psi_{1n}\theta_{1n}^2\}\right], \tag{E.32}$$

$$X_{12} = \left[(\theta_{1n})\ \{\frac{2\nu\Omega}{1-\nu} + n^2\} + \nu\{\theta_{1n}(\psi_{1n}^2-\theta_{1n}^2) + 2\theta_{1n}\psi_{1n}^2\}\right], \qquad (E.33)$$

$$X_{21} = \left[(\psi_{2n})\ \{\frac{2\nu\Omega}{1-\nu^2} + n^2\} + \nu\{(\psi_{2n})(\psi_{2n}^2-\theta_{2n}^2) - 2\psi_{2n}\theta_{2n}^2\}\right], \qquad (E.34)$$

$$X_{22} = \left[(\theta_{2n})\ \{\frac{2\nu\Omega}{1-\nu} + n^2\} + \nu\{\theta_{2n}(\psi_{2n}^2-\theta_{2n}^2) + 2\theta_{2n}\psi_{2n}^2\}\right], \qquad (E.35)$$

$$X_{31} = \left[(\psi_{3n})\ \{\frac{2\nu\Omega}{(1-\nu^2)} + n^2\} + \nu(\psi_{3n})(\psi_{3n}^2-\theta_{3n}^2) - 2\psi_{3n}\theta_{3n}\}\right] \qquad (E.36)$$

$$X_{32} = \left[(\theta_{3n})\ \{\frac{2\nu\Omega}{1-\nu} + n^2\} + \nu\{\theta_{3n}(\psi_{3n}^2-\theta_{3n}^2) + 2\theta_{3n}\psi_{3n}^2\}\right], \qquad (E.37)$$

$$X_{41} = \left[(\psi_{4n})\ \{\frac{2\nu\Omega}{1-\nu^2} + n^2\} + \nu\{(\psi_{4n})(\psi_{4n}^2-\theta_{4n}^2) - 2\psi_{4n}\theta_{4n}^2\}\right], \qquad (E.38)$$

$$X_{42} = \left[(\theta_{4n})\ \{\frac{2\nu\Omega}{1-\nu} + n^2\} + \nu\{\theta_{4n}(\psi_{4n}^2-\theta_{4n}^2) + 2\theta_{4n}\psi_{4n}^2\}\right]\ . \qquad (E.39)$$

APPENDIX F

FUNCTIONS IN PARTICULAR SOLUTION OF DONNELL'S EQUATIONS

This appendix contains expressions for the functions g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, and g_9 which occur in the particular solutions of the Donnell's equations. These functions are given as follows:

$$g_1 = \frac{N_1}{D_1}, \tag{F.1}$$

where

$$N_1 = -(1+\nu)(3-\nu)(1-\nu^2)\rho\alpha_{mn}^2(12Ea^4)(\omega^2) +$$

$$+ (24)(1+\nu)(1-\nu^2)\rho^2 a^4\omega^4 + (1-\nu^2)12E^2a^4\alpha_{mn}^4,$$

and

$$D_1 = \omega^6\{-24(1+\nu)(1-\nu^2)^2\rho^3ha^4\} + \omega^4\{24(1+\nu)(1-\nu^2)\rho^2Eha^2 +$$

$$+ (1+\nu)(1-\nu^2)\rho^2h^2\alpha_{mn}^4(2)Eha^4 + (1+\nu)(3-\nu)(1-\nu^2)\rho^2\alpha_{mn}^2(12)Eha^4\} +$$

$$\omega^2\{-(1+\nu)(3-\nu)\rho\alpha_{mn}^2(12)E^2ha^2 - (1+\nu)(3-\nu)\rho h^2\alpha_{mn}^6E^2ha^4$$

$$- 12(1+\nu)(1-\nu^2)\rho E^2ha^4\alpha_{mn}^4 + 24(1+\nu)(\nu^2)\rho E^2ha^2\alpha_m^2 + 24(1+\nu)\rho E^2hn^2\}$$

$$+ \{E^3h^3a^4\alpha_{mn}^8 + 12(1-\nu^2)E^3a^2h\alpha_m^4\},$$

$$g_2 = \frac{N_2}{(1-\nu)D_1}, \tag{F.2}$$

where

$$N_2 = -24(1+\nu)(1-\nu^2)(1-\nu)Ea^2\rho n(\omega^2) +$$

$$+ n^3(12)E^2(1-\nu^2)(1-\nu) + (2+\nu)(1-\nu^2)^2(\alpha_m^2)(n)12E^2a^2,$$

$$g_3 = \frac{N_3}{D_3}, \tag{F.3}$$

where

$$N_3 = (2+\nu)(\alpha_m^2)(n)g_1E^2a^2 + g_1n^3E^2 - 2(1+\nu)\rho g_1na^2E,$$

and

$$D_3 = E^2a^4\alpha_{mn}^4 - 2(1+\nu)(1-\nu^2)\rho a^4\omega^2 - Ea^4(1+\nu)(3-\nu)\rho\alpha_{mn}^2\omega^2,$$

$$g_4 = \frac{N_4}{D_4}, \tag{F.4}$$

where

$$N_4 = \omega^2\{2(1+\nu)(1-\nu^2)(1-\nu)\rho a^4 - 2(1+\nu)(1-\nu)(n)g_2\rho Eh\} +$$

$$(2+\nu)(\alpha_m^2)(n)g_2E^2h(1-\nu)a^2 + n^3g_2E^2h(1-\nu) - (1-\nu^2)(n^2)E(1-\nu)a^2$$

$$- 2(1-\nu^2)\alpha_m^2Ea^4,$$

and

$$D_4 = E^2h(1-\nu)a^4\alpha_{mn}^4 - 2(1+\nu)(1-\nu^2)(1-\nu)\rho^2ha^4\omega^4$$

$$- 2(1+\nu)\rho\alpha_{mn}^2Eh\ (1-\nu)a^4\omega^2,$$

$$g_5 = \frac{N_5}{D_5}\ , \tag{F.5}$$

where

$$N_5 = Ea^2\nu\alpha_m^3g_1 - 2Ea(1+\nu)\rho\nu\alpha_mg_1\omega^2 - E^2\alpha_m(n^2)g_1,$$

and

$$D_5 = E^2a^2\alpha_{mn}^4 + 2(1+\nu)(1-\nu^2)a^2\rho^2\omega^4 - Ea^2(1+\nu)(3-\nu)\rho\alpha_{mn}^2\omega^2,$$

$$g_6 = \frac{N_6}{D_6}\ , \tag{F.6}$$

where

$$N_6 = E^2a\nu(\alpha_m^3)g_2 - E^2(\alpha_m)(n^2)g_2 - 2Ea(1+\nu)\rho\alpha_mg_2\nu,$$

and

$$D_6 = E^2a^2\alpha_{mn}^4 + 2a^2(1+\nu)(1-\nu)\rho^2\omega^4 - Ea^2(1+\nu)(3-\nu)(\alpha_{mn}^2)\rho\omega^2,$$

$$g_7 = \frac{N_7}{D_1}\ , \tag{F.7}$$

where

$$N_7 = 2\rho\nu(1-\nu^2)\alpha_m Ea^3(\omega^2) - \nu(1-\nu^2)\alpha_m^3E^2a^3 - (1-\nu^2)(\alpha_m)n^2)E^2a$$

$$g_8 = \frac{N_8}{D_8}\ , \tag{F.8}$$

where

$$N_8 = (2+\nu)(\alpha_m^2)(n)E^2a^2h + (n)^3E^2hg_7 + Ea^3(1+\nu)^2(\alpha_m)(n),$$

$$D_8 = E^2a^4h\alpha_{mn}^4 + 2(1+\nu)(1-\nu^2)a^4h\rho^2\omega^4 - Ea^4h(1+\nu)(3-\nu)\rho\omega^2\alpha_{mn}^2,$$

$$g_9 = \frac{N_9}{D_9}\ , \tag{F.9}$$

where

$$N_9 = g_7\nu(\alpha_m^3)E^2ha - g_7\alpha_m(n^2)E^2a^2h - 2g_7(1+\nu)\nu\rho\omega^2\alpha_m Ea^3h$$

$$+ 2(1+\nu)(1-\nu^2)\rho\omega^2a^2 - (1-\nu^2)\alpha_m^2Ea^2 - 2(1+\nu)(n^2)E,$$

and

$$D_9 = E^2ha^2\alpha_{mn}^4 + 2(1+\nu)(1-\nu^2)^2\rho\omega^4ha^2 - Eha^2(1+\nu)(3-\nu)\rho\alpha_{mn}^2\omega^2,$$

and

$$\alpha_{mn}^4 = \left\{\left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n}{a}\right)^2\right\}^2 .$$

APPENDIX G

ANALYSIS FOR SIMPLY SUPPORTED STIFFENED CYLINDRICAL SHELLS

For the special case when both edges of the shell are simply supported, it is not necessary to determine an additional complementary solution since it is possible to find particular solutions which will satisfy all of the boundary conditions as well as the differential equations. Adopting the notation of Chapter IV, the general solution of the shell equations is given by:

$$u(x,\phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_5 Z_{jmn} + G_5 T_{jmn} + g_6 Y_{jmn} + g_9 X_{jmn}) \cos n\phi \cos \frac{m\pi x}{\ell} +$$

$$\sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_5 Z'_{jmn} + g_5 T'_{jmn} + g_6 Y'_{jmn} + g_9 X'_{jmn}) \sin n\phi \cos \frac{m\pi x}{\ell} , \tag{G.1}$$

$$v(x,\phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_3 Z_{jmn} + g_3 T_{jmn} + g_4 Y_{jmn} + g_8 X_{jmn}) \sin n\phi \sin \frac{m\pi x}{\ell} -$$

$$\sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_3 Z'_{jmn} + g_3 T'_{jmn} + g_4 Y'_{jmn} + g_8 X'_{jmn}) \cos n\phi \sin \frac{m\pi x}{\ell} , \tag{G.2}$$

$$w(x,\phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_1 Z_{jmn} + g_1 T_{jmn} + g_2 Y_{jmn} + g_7 X_{jmn}) \cos n\phi \sin \frac{m\pi x}{\ell} +$$

$$\sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_1 Z'_{jmn} + g_1 T'_{jmn} + g_2 Y'_{jmn} + g_7 X'_{jmn}) \sin n\phi \sin \frac{m\pi x}{\ell} . \tag{G.3}$$

The general solutions of the stringer equations are given by:

$$u_{oj}(x) = \sum_{m=0}^{\infty} \zeta_{4m} X_{jm} \cos\left(\frac{m\pi x}{\ell}\right), \tag{G.4}$$

$$v_{oj}(x) = \sum_{m=1}^{\infty} \zeta_{om} Y_{jm} \sin\left(\frac{m\pi x}{\ell}\right), \tag{G.5}$$

$$w_{oj}(x) = \sum_{m=1}^{\infty} \zeta_{1m} Z_{jm} \sin\left(\frac{m\pi x}{\ell}\right), \tag{G.6}$$

$$\theta_{oj}(x) = \sum_{m=1}^{\infty} \zeta_{2m} T_{jm} \sin\left(\frac{m\pi x}{\ell}\right) - z_j \sum_{m=1}^{\infty} \zeta_{2m} Y_{jm} \sin\left(\frac{m\pi x}{\ell}\right). \tag{G.7}$$

The kinematical boundary conditions at the line of attachment are given by:

$$w_{oi}(x) = w(x,\phi_i), \quad (i=1,2,\ldots,K) \tag{G.8}$$

$$\theta_{oi}(x) = \frac{1}{a} \left.\frac{\partial w}{\partial \phi}\right|_{\phi=\phi_i}, \quad (i=1,2,\ldots,K) \tag{G.9}$$

$$v_{oi}(x) = v(x,\phi_i) + \left(\frac{z_i}{a}\right) \left.\frac{\partial w}{\partial \phi}\right|_{\phi=\phi_i}, \quad (i=1,2,\ldots,K) \tag{G.10}$$

$$u_{oi}(x) = u(x,\phi_i), \quad (i=1,2,\ldots,K) \tag{G.11}$$

where K is the total number of stringers attached to the shell.

Inserting Equations (G.1), (G.2), (G.3), and (G.4), (G.5), (G.6), (G.7) into (G.8), (G.9), (G.10), and (G.11) gives the following

Inserting Equations (G.1), (G.2), (G.3), and (G.4), (G.5), (G.6), (G.7) into (G.8), (G.9), (G.10), and (G.11) gives the following matrix equation:

$$\left[\begin{array}{c|c|c|c} A_{ip} & B_{ip} & C_{ip} & K_{ip} \\ \hline E_{ip} & F_{ip} & G_{ip} & H_{ip} \\ \hline I_{ip} & J_{ip} & L_{ip} & M_{ip} \\ \hline N_{ip} & O_{ip} & R_{ip} & S_{ip} \end{array}\right] \left(\begin{array}{c} \bar{Z}_p \\ \hline \bar{T}_p \\ \hline \bar{Y}_p \\ \hline \bar{X}_p \end{array}\right) = \left(\begin{array}{c} \bar{0} \\ \hline \bar{0} \\ \hline \bar{0} \\ \hline \bar{0} \end{array}\right), \tag{G.12}$$

where $\bar{Z}_p$, $\bar{T}_p$, $\bar{Y}_p$, and $\bar{X}_p$ are column vectors whose components are given by:

$$\bar{Z}_p = Z_{jm}, \tag{G.13}$$

$$\bar{T}_p = T_{jm}, \tag{G.14}$$

$$\bar{Y}_p = Y_{jm}, \tag{G.15}$$

$$\bar{X}_p = X_{jm}, \tag{G.16}$$

and m is a fixed integer (axial wave number) and j is related to P by:

$$j = P - K[(P-1)/K]_T \tag{G.17}$$

and the subscript T represents the operation of truncation after the decimal point. The coefficients A_{ip}, B_{ip}, etc. are given by:

$$A_{ip} = A_{ij} = \sum_{n=0}^{N^*} [g_1 f_1(\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)] - \delta_{ij} Z_{jm}, \tag{G.18}$$

$$B_{ip} = B_{ij} = \sum_{n=0}^{N^*} [g_1 f_2(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)], \tag{G.19}$$

$$C_{ip} = C_{ij} = \sum_{n=1}^{N^*} [g_4 f_3(\sin n\phi_j \cos n\phi_i + \cos n\phi_j \sin n\phi_i)], \tag{G.20}$$

$$K_{ip} = K_{ij} = \sum_{n=0}^{N^*} [g_7 f_4(\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)], \tag{G.21}$$

$$E_{ip} = E_{ij} = \sum_{n=0}^{N^*} [g_1 f_1(\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)(n/a)], \tag{G.22}$$

$$F_{ip} = F_{ij} = \sum_{n=0}^{N^*} [-g_1 f_2(\cos n\phi_j \sin n\phi_i + \sin n\phi_j \cos n\phi_i)]$$

$$(n/a) - \delta_{ij} \zeta_{2m}, \tag{G.23}$$

$$G_{ip} = G_{ij} = \sum_{n=0}^{N^*} [g_2 f_3(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i)]$$

$$(n/a) - \delta_{ij} z_j \zeta_{2m}, \tag{G.24}$$

$$H_{ip} = H_{ij} = -\sum_{n=0}^{N^*} [g_7 f_4(\sin n\phi_j \cos n\phi_i + \cos n\phi_j \sin n\phi_i)](n/a), \tag{G.25}$$

$$I_{ip} = I_{ij} = \sum_{n=0}^{N^*}[g_3 f_1(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)] -$$

$$\sum_{n=0}^{N^*}[g_1 f_1(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)]\left(\frac{z_i}{a}\right)(n), \tag{G.26}$$

$$J_{ip} = J_{ij} = \sum_{n=0}^{N^*}[g_3 f_2(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i)] -$$

$$\sum_{n=0}^{N^*}[g_1 f_2(\cos n\phi_j \sin n\phi_i + \sin n\phi_j \cos n\phi_i)]\left(\frac{z_i}{a}\right)(n), \tag{G.27}$$

$$L_{ip} = L_{ij} = \sum_{n=0}^{N^*}[g_4 f_3(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i)] -$$

$$\sum_{n=0}^{N^*}[g_2 f_3(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i)]$$

$$\left(\frac{z_i}{a}\right)(n) - \delta_{ij}\zeta_{om}, \tag{G.28}$$

$$M_{ip} = M_{ij} = \sum_{n=0}^{N^*}[g_9 f_4(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i)] -$$

$$\sum_{n=0}^{N^*}[g_7 f_4(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i)]\left(\frac{z_i}{a}\right)n, \tag{G.29}$$

$$N_{ip} = N_{ij} = \sum_{n=0}^{N^*} [g_5 f_1 (\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)], \quad (G.30)$$

$$O_{ip} = O_{ij} = \sum_{n=0}^{N^*} [g_5 f_2 (\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i)], \quad (G.31)$$

$$R_{ip} = R_{ij} = \sum_{n=0}^{N^*} [g_6 f_3 (\sin n\phi_j \cos n\phi_i + \cos n\phi_j \sin n\phi_i)], \quad (G.32)$$

$$S_{ip} = S_{ij} = \sum_{n=0}^{N^*} [g_9 f_4 (\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i)]$$

$$- \zeta_{4m} \delta_{ij}, \quad (G.33)$$

where i denotes the ith row.

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